AN

INTRODUCTION

TO

ALGEBRA,

BEING THE

FIRST PART

OF A

COURSE OF MATHEMATICS,

ADAPTED

TO THE METHOD OF INSTRUCTION

IN THE

AMERICAN COLLEGES.

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PREFACE.

The following summary view of the first principles of algebra is intended to be accommodated to the method of instruction generally adopted in the American colleges.

The books which have been published in Great Britain on mathematical subjects, are principally of two classes.—One consists of extended treatises, which enter into a thorough investigation of the particular departments which are the objects of their inquiry. Many of these are excellent in their kind; but they are too voluminous for the use of the body of students in a college.

The other class are expressly intended for beginners; but many of them are written in so concise a manner, that important proofs and illustrations are excluded. They are mere *text-books*, containing only the *outlines* of subjects which are to be explained and enlarged upon, by the professor in his lecture room, or by the private tutor in his chamber.

In the colleges in this country, there is generally put into the hands of a class, a book from which they are expected *of themselves* to acquire the principles of the science to which they are attending: receiving, however, from their instructor, any additional assistance which may be found necessary. An elementary work for such a purpose, ought evidently to contain the explanations which are requisite, to bring the subjects treated of within the comprehension of the body of the class.

If the design of studying the mathematics were merely to obtain such a knowledge of the *practical* parts, as is required for transacting business; it might be sufficient to commit to memory some of the principal rules, and to make the operations familiar, by attending to the examples. In this mechanical way, the accountant, the navigator, and the land surveyor, may be qualified for their respective employments, with very little knowledge of the *principles* that lie at the foundation of the calculations which they are to make.

But a higher object is proposed, in the case of those who are acquiring a liberal education. The main design should be to call into exercise, to discipline, and to invigorate the powers of the mind. It is the *logic* of the mathematics which

In the selection of materials, those articles have been taken which have a practical application, and which are preparatory to succeeding parts of the mathematics, philosophy, and astronomy. The object has not been to introduce *original matter*. In the mathematics, which have been cultivated with success from the days of Pythagoras, and in which the principles already established are sufficient to occupy the most active mind for years, the parts to which the student ought *first* to attend, are not those recently discovered. Free use has been made of the works of Newton, Maclaurin, Saunderson, Simpson, Euler, Emerson, Lacroix, and others, but in a way that rendered it inconvenient to refer to them, in particular instances. The proper field for the display of mathematical *genius*, is in the region of invention. But what is requisite for an elementary work, is to collect, arrange and illustrate, materials already provided. However humble this employment, he ought patiently to submit to it, whose object is to instruct, not those who have made considerable progress in the mathematics, but those who are just commencing the study. Original discoveries are not for the benefit of *beginners*, though they may be of great importance to the advancement of science.

The arrangement of the parts is such, that the explanation of one is not made to depend on another which is to follow. In the statement of general rules, if they are reduced to a small number, their applications to particular cases may not, always, be readily understood. On the other hand, if they are very numerous, they become tedious and burdensome to the memory. The rules given in this introduction, are most of them comprehensive; but they are explained and applied, in subordinate articles.

A *particular* demonstration is sometimes substituted for a *general* one, when the application of the principle to other cases is obvious. The examples are not often taken from philosophical subjects, as the learner is supposed to be familiar with none of the sciences except arithmetic. In treating of *negative* quantities, frequent references are made to mercantile concerns, to debt, and credit, &c. These are merely for the purpose of illustration. The whole doctrine of negatives is made to depend on the single principle, that they are quantities to be *subtracted*. But the student, at this early period, is not accustomed to abstraction. He requires particular examples, to catch his attention, and aid his conceptions.

a demonstration may be safely omitted, when it is so simple and obvious, that no one possessing a moderate acquaintance with the subject, could fail to supply it for himself. But this liberty of omission ought not to be extended to cases in which it will occasion obscurity and embarrassment. If it be desirable to give opportunity for the mind to display and enlarge its powers, by surmounting obstacles; full scope may be found for this kind of exercise, especially in the higher branches of the mathematics, from difficulties which will unavoidably occur, without creating new ones for the sake of perplexing.

Algebra requires to be treated in a more plain and diffuse manner, than some other parts of the mathematics; because it is to be attended to, *early* in the course, while the mind of the learner has not been habituated to a mode of thinking so abstract, as that which will now become necessary. He has also a *new language* to learn, at the same time he is settling the *principles* upon which his future inquiries are to be conducted. These principles ought to be established, in the most clear and satisfactory manner which the nature of the case will admit of. Algebra and geometry may be considered as lying at the foundation of the succeeding branches of the mathematics, both pure and mixed. Euclid and others have given to the geometrical part a degree of clearness and precision which would be very desirable, but is hardly to be expected, in algebra.

For the reasons which have been mentioned, the manner in which the following pages are written, is not the most concise. But the work is necessarily limited in extent of subject. It is far from being a *complete* treatise of algebra. It is merely an introduction. It is intended to contain as much matter, as the student at college can attend to, with advantage, during the short time allotted to this particular study. There is generally but a small portion of a class, who have either leisure or inclination, to pursue mathematical inquiries much farther than is necessary to maintain an honorable standing in the institution of which they are members. Those few who have an unusual taste for this science, and aim to become adepts in it, ought to be referred to separate and complete treatises, on the different branches. No one who wishes to be thoroughly versed in mathematics, should look to compendiums and elementary books for any thing more than the first principles. As soon as these are acquired, he should be guided in his inquiries by the genius and spirit of original authors.

constitutes their principal value, as a part of a course of collegiate instruction. The time and attention devoted to them, is for the purpose of forming *sound reasoners*, rather than expert mathematicians. To accomplish this object it is necessary that the principles be clearly explained and demonstrated, and that the several parts be arranged in such a manner, as to show the dependence of one upon another. The whole should be so conducted, as to keep the reasoning powers in continual exercise, without greatly fatiguing them. No other subject affords a better opportunity for exemplifying the rules of correct thinking. A more finished specimen of clear and exact logic has, perhaps, never been produced, than the Elements of Geometry by Euclid.

It may be thought, by some, to be unwise to form our general habits of arguing, on the model of a science in which the inquiries are accompanied with *absolute certainty;* while the common business of life must be conducted upon *probable* evidence, and not upon principles which admit of complete demonstration. There would be weight in this objection, if the attention were confined to the *pure* mathematics. But when these are connected with the *physical* sciences, astronomy, chemistry, and natural philosophy, the mind has opportunity to exercise its judgment upon all the various degrees of probability which occur in the concerns of life.

So far as it is desirable to form a *taste* for mathematical studies, it is important that the books by which the student is first introduced to an acquaintance with these subjects, should not be rendered obscure and forbidding by their conciseness. Here is no opportunity to awaken interest, by rhetorical elegance, by exciting the passions, or by presenting images to the imagination. The beauty of the mathematics depends on the distinctness of the objects of inquiry, the symmetry of their relations, the luminous nature of the arguments, and the certainty of the conclusions. But how is this beauty to be perceived, in a work which is so much abridged, that the chain of reasoning is often interrupted, important demonstrations omitted, and the transitions from one subject to another so abrupt, as to keep their connections and dependencies out of view?

It may not be necessary to state every proposition and its proof, with all the formality which is so strictly adhered to by Euclid; as it is not essential to a logical argument, that it be expressed in regular and entire syllogisms. A step of

The section on *proportion*, will, perhaps, be thought useless to those who read the fifth Book of Euclid. That is sufficient for the purposes of pure *geometrical* demonstration. But it is important that the propositions should also be presented under the algebraic forms. In addition to this, great assistance may be derived from the algebraic *notation*, in demonstrating, and reducing to system, the laws of proportion. The subject instead of being broken up into a multitude of distinct propositions, may be comprehended in a few general principles.

THE REVISED EDITION.

When it was found necessary to renew the stereotype plates for the algebra, which were too much impaired to be longer used, the opportunity was embraced to make additions and alterations, to adapt it to the advance which had been made, in this department of collegiate instruction, since the work was first written. Professor STANLEY of Yale College was applied to, to make the proposed revision. He had proceeded through the section on simple equations, when it was deemed expedient, that he should suspend his professional engagements, and cross the Atlantic, for the recovery of his health. In his absence, the revision was continued by the author. After his return, Mr. Stanley made the important additions, in the two sections on the general properties and solution of the higher equations. It is not practicable to comprise, in a single volume of moderate size, the entire science and art of algebra. The extent of a text-book on the subject must be proportioned to the amount of time which can be allotted to the study of it, without encroaching upon other departments of instruction, in the colleges, and other scientific institutions.

Some of the additions which have been made, in the present edition, are multiplication and division by detached coefficients, general properties of quadratic equations, permutations and combinations, demonstrations of the binomial theorem, in the four cases of integral, fractional, positive and negative exponents, continued fractions, interpolation, general properties and transformation of equations, Sturm's Theorem, and Horner's Method, for the solution of the higher equations. The last two improvements had not been made known, when this work was originally published.

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INTRODUCTORY OBSERVATIONS

ON THE

MATHEMATICS IN GENERAL.

ART. 1. MATHEMATICS *is the science of* QUANTITY.

Any thing which can be *multiplied*, *divided*, or *measured*, is called *quantity*. Thus, a *line* is a quantity, because it can be doubled, trebled, or halved; and can be measured, by applying to it another line, as a foot, a yard, or an ell. *Weight* is a quantity, which can be measured, in pounds, ounces, and grains. *Time* is a species of quantity, whose measure can be expressed, in hours, minutes, and seconds. But *color* is not a quantity. It can not be said, with propriety, that one color is twice as great, or half as great, as another. The operations of the *mind*, such as thought, choice, desire, hatred, &c. are not quantities. They are incapable of mensuration.*

Those parts of the Mathematics, on which all the others are founded, are *Arithmetic*, *Algebra*, and *Geometry*.

2. ARITHMETIC is the science of *numbers*. Its aid is required to complete and apply the calculations, in almost every other department of the mathematics.

3. ALGEBRA is a method of computing by *letters* and other symbols. *Fluxions*, or the *Differential* and *Integral Calculus*, may be considered as belonging to the higher branches of Algebra.

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* See Note A.

4. GEOMETRY is that part of the mathematics, which treats of *magnitude.* By magnitude, in the appropriate sense of the term, is meant that species of quantity, which is *extended;* that is, which has one or more of the three dimensions, *length, breadth,* and *thickness.* Thus a *line* is a magnitude, because it is extended, in length. A *surface* is a magnitude, having length and breadth. A *solid* is a magnitude, having length, breadth, and thickness. But *motion,* though a quantity, is not, strictly speaking, a magnitude. It has neither length, breadth, nor thickness.*

Trigonometry and *Conic Sections* are branches of the mathematics, in which the principles of Geometry are applied to *triangles,* and the sections of a *cone.*

5. Mathematics are either pure or mixed. In *pure* mathematics, quantities are considered, independently of any substances actually existing. But, in *mixed* mathematics, the relations of quantities are investigated, in connection with some of the properties of matter, or with reference to the common transactions of business. Thus, in Surveying, mathematical principles are applied to the measuring of land; in Optics, to the properties of light; and in Astronomy, to the motions of the heavenly bodies.

The science of the pure mathematics has long been distinguished, for the clearness and distinctness of its principles; and the irresistible conviction, which they carry to the mind of every one who is once made acquainted with them. This is to be ascribed, partly to the nature of the subjects, and partly to the exactness of the definitions, the axioms, and the demonstrations.

6. The foundation of all mathematical knowledge must be laid in definitions and self-evident truths. A *definition* is an explanation of what is meant, by any word or phrase. Thus, an equilateral triangle is defined, by saying, that it is a figure bounded by three equal sides.

It is essential to a complete definition, that it perfectly distinguish the thing defined, from every thing else. On many subjects it is difficult to give such precision to language, that it shall convey, to every hearer or reader, exactly the same ideas. But in the mathematics, the principal terms may be so defined, as not to leave room for the least difference of

* Some writers, however, use magnitude as synonymous with quantity.

apprehension, respecting their meaning. All must be agreed, as to the nature of a circle, a square, and a triangle, when they have once learned the definitions of these figures.

Under the head of definitions, may be included explanations of the *characters* which are used to denote the relations of quantities. Thus the character $\surd$ is explained or defined, by saying that it signifies the same as the words square root.

7. The next step, after becoming acquainted with the meaning of mathematical terms, is to bring them together, in the form of propositions. Some of the relations of quantities require no process of reasoning, to render them evident. To be understood, they need only to be proposed. That a square is a different figure from a circle; that the whole of a thing is greater than one of its parts; and that two straight lines can not enclose a space, are propositions so manifestly true, that no reasoning upon them could make them more certain. They are, therefore, called self-evident truths, or *axioms.*

8. There are, however, comparatively few mathematical truths which are self-evident. Most require to be proved by a chain of reasoning. Propositions of this nature are denominated *theorems;* and the process, by which they are shown to be true, is called *demonstration.* This is a mode of arguing, in which, every inference is immediately derived, either from definitions, from axioms, or from principles which have been previously demonstrated. In this way, complete certainty is made to accompany every step, in a long course of reasoning.

9. Demonstration is either *direct* or *indirect.* The former is the common, obvious mode of conducting a demonstrative argument. But in some instances, it is necessary to resort to indirect demonstration; which is a method of establishing a proposition, by proving that to suppose it *not* true, would lead to an absurdity. This is frequently called *reductio ad absurdum.* Thus, in certain cases in geometry, two lines may be proved to be equal, by showing that to suppose them unequal, would involve an absurdity.

10. Besides the principal theorems in the mathematics, there are also Lemmas and Corollaries.

A *lemma* is a proposition which is demonstrated, for the purpose of using it, in the demonstration of some other proposition. This preparatory step is taken to prevent the proof

of the principal theorem from becoming complicated and tedious.

A *corollary* is an inference from a preceding proposition. A Scholium is a remark of any kind, suggested by something which has gone before, though not, like a corollary, immediately depending on it.

11. The immediate object of inquiry, in the mathematics, is, frequently, not the demonstration of a general truth, but a method of performing some operation, such as reducing a vulgar fraction to a decimal, extracting the cube root, or inscribing a circle in a square. This is called solving a problem. A *theorem* is something to be *proved*. A *problem* is something to be *done*.

When that which is required to be done, is so easy, as to be obvious to every one, without an explanation, it is called a *postulate*. Of this nature is the drawing of a straight line, from one point to another.

12. A quantity is said to be *given*, when it is either supposed to be already *known*, or is made a *condition*, in the statement of any theorem or problem. In the rule of proportion in arithmetic, for instance, three terms must be given to enable us to find a fourth. These three terms are the *data*, upon which the calculation is founded. If we are required to find the number of acres, in a circular island ten miles in circumference, the circular figure, and the length of the circumference are the data. They are said to be given *by supposition*, that is, by the conditions of the problem. A quantity is also said to be given, when it may be directly and easily *inferred* from something else which is given. Thus, if two numbers are given, their *sum* is given; because it is obtained, by merely adding the numbers together.

In Geometry, a quantity may be given, either in *position*, or *magnitude*, or both. A line is given in position, when its *situation* and *direction* are known. It is given in magnitude, when its *length* is known. A circle is given in *position*, when the place of its centre is known. It is given in *magnitude*, when the length of its diameter is known.

13. One proposition is *contrary* or contradictory to another, when, what is affirmed, in the one, is denied, in the other.

A proposition and its contrary, can never *both* be true. It can not be true, that two given lines are equal, and that they are *not* equal, at the same time.

14. One proposition is the *converse* of another, when the order is inverted; so that, what is *given* or supposed in the first, becomes the *conclusion* in the last; and what is given in the last, is the conclusion, in the first. Thus, it can be proved, first, that if the *sides* of a triangle are equal, the *angles* are equal; and secondly, that if the *angles* are equal, the *sides* are equal. Here, in the first proposition, the equality of the *sides is given;* and the equality of the *angles inferred:* in the second, the equality of the *angles* is given, and the equality of the *sides* inferred.

In many instances, a proposition and its converse are both true; as in the preceding example. But this is not always the case. A circle is a figure bounded by a curve; but a figure bounded by a curve is not of course a circle.

15. The practical applications of the mathematics, in the common concerns of business, in the useful arts, and in the various branches of physical science are almost innumerable. Mathematical principles are necessary in *Mercantile transactions*, for keeping, arranging, and settling accounts, adjusting the prices of commodities, and calculating the profits of trade: in *Navigation*, for directing the course of a ship on the ocean, adapting the position of her sails to the direction of the wind, finding her latitude and longitude, and determining the bearings and distances of objects on shore: in *Surveying*, for measuring, dividing, and laying out grounds, taking the elevation of hills, and fixing the boundaries of fields, estates and public territories: in *Civil Engineering*, for constructing bridges, aqueducts, locks, &c.: in *Mechanics*, for understanding the laws of motion, the composition of forces, the equilibrium of the mechanical powers, and the structure of machines: in *Architecture*, for calculating the comparative strength of timbers, the pressure which each will be required to sustain, the forms of arches, the proportions of columns, &c.: in *Fortification*, for adjusting the position, lines, and angles, of the several parts of the works: in *Gunnery*, for regulating the elevation of the cannon, the force of the powder, and the velocity and range of the shot: in *Optics*, for tracing the direction of the rays of light, understanding the formation of images, the laws of vision, the separation of colors, the nature of the rainbow, and the construction of microscopes and telescopes: in *Astronomy*, for computing the distances, magnitudes, and revolutions of the heavenly bodies; and the influence of the law of gravitation, in raising the tides, dis-

turbing the motions of the moon, causing the return of the comets, and retaining the planets in their orbits: in *Geography*, for determining the figure and dimensions of the earth, the extent of oceans, islands, continents, and countries; the latitude and longitude of places, the courses of rivers, the height of mountains, and the boundaries of kingdoms: in *History*, for fixing the chronology of remarkable events, and estimating the strength of armies, the wealth of nations, the value of their revenues, and the amount of their population: and, in the concerns of *Government*, for apportioning taxes, arranging schemes of finance, and regulating national expenses. The mathematics have also important applications to Chemistry, Mineralogy, Music, Painting, Sculpture, and indeed to a great proportion of the whole circle of arts and sciences.

16. It is true, that, in many of the branches which have been mentioned, the ordinary business is frequently transacted, and the mechanical operations performed, by persons who have not been regularly instructed in a course of mathematics. Machines are framed, lands are surveyed, and ships are steered, by men who have never thoroughly investigated the principles, which lie at the foundation of their respective arts. The reason of this is that the methods of proceeding, in their several occupations, have been pointed out to them, by the genius and labor of others. The mechanic often works by rules, which men of science, have provided for his use, and of which he knows nothing more than the practical application. The mariner calculates his longitude by tables, for which he is indebted to mathematicians and astronomers of no ordinary attainments. In this manner, even the abstruse parts of the mathematics, are made to contribute their aid to the common arts of life.

17. But an additional and more important advantage to persons of liberal education, is to be found, in the enlargement and improvement of the reasoning powers. The mind, like the body, acquires strength by exertion. The art of reasoning, like other arts, is learned by practice. It is perfected, only by long continued exercise. Mathematical studies are peculiarly fitted for this discipline of the mind. They are calculated to form it to habits of fixed attention; of sagacity, in detecting sophistry; of caution, in the admission of proof; of dexterity in the arrangement of arguments; and

of skill, in making all the parts of a long continued process tend to a result, in which the truth is clearly and firmly established. When a habit of close and accurate thinking is thus acquired, it may be applied to any subject, on which a man of letters or of business may be called to employ his talents. "The youth," says Plato, "who are furnished with mathematical knowledge, are prompt and quick, at all other sciences."

It is not pretended, that an attention to other objects of inquiry is rendered unnecessary, by the study of the mathematics. It is not their office, to lay before us historical facts; to teach the principles of morals; to store the fancy with brilliant images; or to enable us to speak and write with rhetorical vigor and elegance. The beneficial effects which they produce on the mind, are to be seen, principally, in the regulation and increased energy of the *reasoning powers*. These they are calculated to call into frequent and vigorous exercise. At the same time, mathematical studies may be so conducted, as not often to require excessive exertion and fatigue. Beginning with the more simple subjects, and ascending gradually to those which are more complicated, the mind acquires strength as it advances; and by a succession of steps, rising regularly one above another, is enabled to surmount the obstacles which lie in its way. In a course of mathematics, the parts succeed each other in such a connected series, that the preceding propositions are preparatory to those which follow. The student who has made himself master of the former, is qualified for a successful investigation of the latter. But he who has passed over any of the ground superficially, will find that the obstructions to his future progress are yet to be removed. In mathematics as in war, it should be made a principle, not to advance, while any thing is left unconquered behind. It is important that the student should be deeply impressed with a conviction of the necessity of this. Neither is it sufficient that he understands the *nature* of one proposition or method of operation, before proceeding to another. He ought also to make himself *familiar* with every step, by careful attention to the examples. He must not expect to become thoroughly versed in the science, by merely *reading* the main principles, rules, and observations. It is practice only, which can put these completely in his possession. The method of studying here recommended, is not only that which promises success, but that which will

be found, in the end, to be the most expeditious, and by far the most pleasant. While a superficial attention occasions perplexity and consequent aversion; a thorough investigation is rewarded with a high degree of gratification. The peculiar entertainment which mathematical studies are calculated to furnish to the mind, is reserved for those who make themselves masters of the subjects to which their attention is called.

NOTE. The principal definitions, theorems, rules, &c. which it is necessary to *commit to memory*, are distinguished by being put in Italics or Capitals.

ALGEBRA.

SECTION I.

NOTATION, DEFINITIONS, NEGATIVE QUANTITIES, AXIOMS, &C.

ART. **18.** ALGEBRA *may be defined*, A BRIEF AND GENERAL METHOD OF SOLVING QUESTIONS CONCERNING NUMBERS, BY MEANS OF LETTERS, AND OTHER SYMBOLS.

This, it must be acknowledged, is an imperfect account of the subject; as every account must necessarily be, which is comprised in the compass of a definition. Its real nature is to be learned, rather by an attentive examination of its parts, than from any summary description.

19. The solutions in Algebra, are of a more *general* nature than those in common Arithmetic. The latter relate to particular numbers; the former to whole *classes* of quantities. On this account, Algebra has been termed a kind of *universal Arithmetic*. The generality of its solutions is principally owing to the use of *letters*, instead of numeral figures, to express the several quantities which are subjected to calculation. One of the nine digits, invariably expresses the same number. The figure 8 always signifies eight; the figure 5, five, &c. And, though one of the digits, in connection with others, may have a *local* value, different from its simple value when alone; yet the same *combination* always expresses the same number. Thus 263 has one uniform signification. And this is the case with every other combination of figures. In Arithmetic, therefore, when a problem is solved, the answer is limited to the particular numbers which are specified, in the statement of the question. But an Algebraic solution may be equally applicable to all other quantities which have the same relations. For in Algebra, a letter may stand for any quantity

which we wish it to represent. Thus b may be put for 2, or 10, or 50, or 1000. It must not be understood from this, however, that the letter has no determinate value. Its value is fixed for the occasion. For the present purpose, it remains unaltered. But on a different occasion, the same letter may be put for any other number.

20. A calculation may also be greatly *abridged* by the use of letters; especially when very large numbers are concerned. When several such numbers are to be combined, as in multiplication, the process becomes extremely tedious. But a single letter may be put for a large number, as well as for a small one. The numbers 26347297, 68347823, and 27462498, for instance, may be expressed by the letters, b, c, and d. The multiplying them together, as will be seen hereafter, will be nothing more than writing them, one after another, in the form of a word, and the product will be simply bcd. This indeed is indicating rather than performing the multiplication. But it is often sufficient thus to indicate what is to be done, without executing the work. It may happen, in the solution of a problem, that a multiplication at one time will be counterbalanced by a subsequent division, and the trouble of performing the two operations will be saved, if the first be only indicated, till its effect is destroyed by the other. Solutions in Algebra are sometimes effected, in the compass of a few lines, which, in common Arithmetic, must be extended through many pages.

21. Another advantage obtained from the notation by letters instead of figures, is, that the several quantities which are brought into calculation, may be preserved *distinct from each other,* though carried through a number of complicated processes; whereas, in Arithmetic, they are so blended together, that no trace is left of what they were, before the operation began.

22. Algebra differs farther from Arithmetic, in making use of *unknown* quantities, in carrying on its operations. In Arithmetic, all the quantities which enter into a calculation must be known. For they are expressed *in numbers.* And every number must necessarily be a determinate quantity. But in Algebra, a letter may be put for a quantity, before its value has been ascertained. And yet it may have such relations to other quantities, with which it is connected, as to answer an important purpose in the calculation.

NOTATION AND DEFINITIONS.

23. To facilitate the investigations in algebra, the several steps of the reasoning, instead of being expressed in *words*, are translated into the language of signs and symbols, which may be considered as a species of *short-hand.* This serves to place the quantities and their relations distinctly before the eye, and to bring them all into view at once. They are thus more readily compared and understood, than when removed at a distance from each other, as in the common mode of writing. But before any one can avail himself of this advantage, he must become perfectly familiar with the new language.

24. The *quantities* in algebra, as has been already observed, are generally expressed by *letters.* The *first* letters of the Alphabet are used to represent *known* quantities; and the *last* letters, those which are *unknown.* Sometimes the known quantities, instead of being expressed by letters, are set down in figures, as in common arithmetic.

25. Besides the letters and figures, there are certain characters used, to indicate the *relations* of the quantities, or the *operations* which are performed with them.

The signs + and −, which are read *plus* and *minus*, or *more* and *less*, are employed to denote addition and subtraction. Thus $a+b$ signifies that b is to be added to a. It is read a plus b, or a added to b, or a and b. If the expression be $a-b$, that is, a minus b; it indicates that b is to be subtracted from a.

26. Quantities having the sign + prefixed are called *positive,* and those which have the sign −, *negative* quantities. For the nature of this distinction, see Art. 49.

All the quantities which enter into an algebraic process, are considered, for the purposes of calculation, as either positive or negative. Before the *first* one, unless it be negative, the sign is generally omitted. But it is always to be understood. Thus $a+b$, is the same as $+a+b$.

27. Sometimes *both* + and − are prefixed to the same letter. The sign is then said to be *ambiguous.* Thus $a \pm b$ signifies that in certain cases, comprehended in a general solution, b is to be added to a, and in other cases subtracted from it.

28. When it is intended to express the difference between two quantities, without deciding which is the one to be subtracted, the character $\backsim$ is used. Thus $a \backsim b$, denotes the difference between a and b, without determining whether a is to be subtracted from b, or b from a.

29. The character $\times$ denotes *multiplication*. Thus $a \times b$ is a multiplied into b; and 6×3 is 6 times 3, or 6 into 3. Sometimes a point is used to indicate multiplication. Thus $a.b$ is the same as $a\times b$. But the multiplication of quantities that are represented by letters, is more commonly indicated by connecting the letters together in the form of a word or syllable. Thus ab is the same as $a.b$ or $a\times b$. And $bcde$ is the same as $b\times c\times d\times e$.

When, however, the quantities are expressed numerically, the sign of multiplication must not be omitted. If 35 were written for the product of 3 into 5, it might be mistaken for *thirty-five*.

30. When two or more quantities are multiplied together, each of them is called a *factor*. In the product ab, a is a factor, and so is b.

A quantity is said to be *resolved into factors*, when any factors are taken, which, being multiplied together, will produce the given quantity. Thus ab may be resolved into the two factors a and b, because $a\times b$ is ab. And amn may be resolved into the three factors a, and m, and n. And 48 may be resolved into the two factors 2×24, or 3×16, or 4×12, or 6×8; or into the three factors $2\times3\times8$, or $4\times6\times2$, &c.

31. A numeral figure is often prefixed to a letter. This is called a *co-efficient*. It shows how often the quantity expressed by the letter is to be taken. Thus $2b$ signifies twice b; and $9b$, 9 times b, or 9 multiplied into b.

The co-efficient may be either a whole number or a fraction. Thus $\frac{2}{3}b$ is two-thirds of b. When the co-efficient is not expressed, 1 is always to be understood. Thus a is the same as $1a$; that is, once a.

The co-efficient may be a *letter*, as well as a figure. In the quantity mb, m may be considered the co-efficient of b; because b is to be taken as many times as there are units in m. If m stands for 6, then mb is 6 times b. In $3abc$, 3 may be considered as the co-efficient of abc; $3a$ the co-efficient of bc; or $3ab$, the co-efficient of c. Every co-efficient is a *factor*. (Art. 30.)

32. The character $\div$ is used to show that the quantity which precedes it, is to be *divided*, by that which follows. Thus $a \div c$ is a divided by c; and $ab \div cd$ is the product of a and b, divided by the product of c and d.

But in algebra, division is more commonly expressed, by writing the divisor under the dividend, in the form of a vulgar fraction. Thus $\frac{a}{b}$ is the same as $a \div b$; and $\frac{c \backsim b}{d+h}$ is the difference of c and b, divided by the sum of d and h.

A character prefixed to the dividing line of a fractional expression, is to be understood as referring to all the parts taken collectively; that is to the whole value of the quotient. Thus $a - \frac{b+c}{m+n}$ signifies that the quotient of $b+c$ divided by $m+n$, is to be subtracted from a. And $\frac{c-d}{a+m} \times \frac{h+n}{x-y}$ denotes that the first quotient is to be multiplied into the second.

33. The *equality* between two quantities or sets of quantities is expressed by parallel lines $=$. Thus $a+b=d$ signifies that a and b together are equal to d. And $a+d=c=b+g=h$, signifies that a and d equal c, which is equal to b and g, which are equal to h. So $8+4=16-4=10+2=7+2+3=12$.

34. The *inequality* of two quantities is indicated by placing the character $>$ between them. Thus $a>b$ signifies that a is greater than b.

If the first quantity is *less* than the other, the character $<$ is used; as $a<b$; that is, a is less than b. In both cases, the quantity towards which the character *opens*, is greater than the other.

35. When four quantities are *proportional*, the proportion is expressed by points, in the same manner, as in the Rule of Three in arithmetic. Thus $a : b :: c : d$ signifies that a has to b, the same ratio which c has to d. And $ab : cd :: a+m : b+n$, means, that ab is to cd; as the sum of a and m, to the sum of b and n.

Three points $\therefore$ are sometimes used to signify *therefore* or *consequently*.

36. A *power* of a quantity is the product formed by multiplying the quantity into itself. Thus 2×2 or 4 is the *square* or the second power of 2; $2 \times 2 \times 2$ or 8 is the *cube* or the third power of 2; and $2 \times 2 \times 2 \times 2$ or 16, the fourth power.

So $a \times a$ or aa is the second power of a; $a \times a \times a$ or aaa, the third power; $aaaa$, the fourth power; &c.

The original quantity itself, though not, like the powers proceeding from it, produced by multiplication, is nevertheless called the *first power*. It is also called the *root* of the other powers, because it is that from which they are all derived.

37. As it is inconvenient, especially in the case of high powers, to write down all the letters or factors of which the powers are composed, an abridged method of notation is generally adopted. The root is written only once; and then a number or letter is placed at the right hand, and a little elevated, to signify how many times the root is *employed as a factor*, to produce the power. This number or letter is called the *index* or *exponent* of the power. Thus a^2 is put for $a \times a$ or aa, because the root a, is *twice* repeated as a factor, to produce the power aa. And a^3 stands for aaa; for here a is repeated *three times* as a factor.

The index of the *first* power is 1; but this is commonly omitted. Thus a^1 is the same as a.

Exponents must not be confounded with *co-efficients*. A co-efficient shows how often a quantity is taken as a *part* of a whole. An exponent shows how often a quantity is taken as a *factor* in a product.

Thus $4a=a+a+a+a$. But $a^4=a \times a \times a \times a$.

38. A *simple* quantity is either a single letter or number, or several letters connected together without the signs + and −. Thus a, ab, abd and $8b$ are each of them simple quantities. A simple quantity is also called a *monomial* or a *term*.

A *compound* quantity consists of a number of terms or simple quantities connected by the sign + or −. Thus $a+b$, $d-y$, $b-d+3h$, are each compound quantities. Compound quantities are often called *polynomials*.

If there are *two* terms in a compound quantity, it is commonly called a *binomial*. Thus $a+b$ and $a-b$ are binomials. The latter is also called a *residual* quantity, because it expresses the difference of two quantities, or the remainder, after one is taken from the other. A compound quantity consisting of *three* terms, is sometimes called a *trinomial;* one of four terms, a *quadrinomial*, &c.

39. *Similar* or *like* terms are those in which the letters are the same, and have the same exponents. And *unlike*

terms are those in which the letters are different, or the same letter has different exponents. Thus a^2b, $3a^2b$, $-a^2b$, and $-6a^2b$ are like terms, because their letters and exponents are the same, although the signs and co-efficients are different. But the terms $3ab$, $3a^2b$, and $3axy$ are unlike, because they have not the same letters and exponents, although there is no difference in the signs and co-efficients.

40. Every letter that occurs as a factor in a term, is called a *dimension* of that term. And the *degree* of a term answers to the number of its dimensions. A numeral co-efficient is not reckoned as a dimension. Thus, $2a$ is a term of one dimension, or of the first degree; $-3ab$ is a term of two dimensions, or of the second degree; and $5ab^3c^2$, which is the same as $5abbbcc$, is a term of six dimensions or of the sixth degree.

The degree of a term, or the number of its dimensions, is marked by the sum of the exponents of the letters contained in the term. Thus, the number of dimensions of the term $6ab^3cd^2$ is $1+3+1+2$, or 7.

A polynomial is said to be *homogeneous*, when all its terms are of the same degree. Thus, $7abc-2x^3+3xy^2$ is homogeneous: but $3ab+2ab^2-4cx$ is *not* homogeneous.

41. When the several members of a compound quantity are to be subjected to the same operation, they are frequently connected by a line called a *vinculum*. Thus $a-\overline{b+c}$ shows that the *sum* of b and c is to be subtracted from a. But $a-b+c$ signifies that b only is to be subtracted from a, while c is to be added. The sum of c and d, subtracted from the sum of a and b, is $\overline{a+b}-\overline{c+d}$. The marks used for parentheses, (), are often substituted instead of a line for a vinculum. Thus $x-(a+c)$ is the same as $x-\overline{a+c}$.

The *equality* of two sets of quantities is expressed, without using a vinculum. Thus $a+b=c+d$ signifies, not that b is equal to c; but that the sum of a and b is equal to the sum of c and d.

42. One quantity is said to be a *multiple* of another, when the former *contains* the latter a certain number of times without a remainder. Thus $10a$ is a multiple of $2a$; and 24 is a multiple of 6.

One quantity is said to be a *measure* of another, when the former is *contained* in the latter, any number of times, with-

out a remainder. Thus $3b$ is a measure of $15b$; and 7 is a measure of 35.

43. *The* RECIPROCAL *of a quantity, is the quotient arising from dividing* A UNIT *by that quantity.* Thus the reciprocal of a is $\frac{1}{a}$; the reciprocal of $a+b$ is $\frac{1}{a+b}$; the reciprocal of 4 is $\frac{1}{4}$.

44. A single letter, or a number of letters, representing any quantities with their relations, is called an algebraic *expression;* and sometimes a *formula.* Thus $a+b+3d$ is an algebraic expression.

The *value* of an expression, is the number or quantity for which the expression stands. Thus the value of $3+4$ is 7; of 3×4 is 12; of $\frac{16}{8}$ is 2.

45. The relations of quantities, which in ordinary language are signified by *words*, are represented in the algebraic notation, by *signs*. The latter mode of expressing these relations ought to be made so familiar to the mathematical student, that he can, at any time, substitute the one for the other.

A few examples are here added, in which, words are to be converted into signs.

1. What is the algebraic expression for the following statement, in which the letters a, b, c, &c. may be supposed to represent any given quantities?

The product of a, b and c, divided by the difference of c and d, is equal to the sum of b and c added to 15 times h.

Ans. $\frac{abc}{c \backsim d}=b+c+15h$.

2. The quotient of a divided by ten times the square of b, diminished by twice the product of c and d, is equal to the quotient of 6 divided by the difference of a and d. Ans.

3. The sum of a, b and c, divided by the fifth power of d, is equal to 4 times the difference of b and c, added to 9 times their product. Ans.

4. The product of a, b and c, is to the quotient of a divided by the difference of b and c, as half the sum of a and h, is to 4 times the difference of c and d. Ans.

5. The difference between twice the cube of a and the product of b into c, is equal to the sum of a and c, added to the quotient of b divided by the sum of d, h and m. Ans.

46. It is necessary also, to be able to reverse what is done in the preceding examples, that is, to translate the algebraic signs into common language.

What will the following expressions become, when words are substituted for the signs.

1. $\frac{a+b}{h}=abc-6m+\frac{a}{a+c}$.

Ans. The sum of a and b divided by h, is equal to the product of a, b and c, diminished by 6 times m, and increased by the quotient of a divided by the sum of a and c.

2. $2(a+m^3)+\frac{n}{6\sim b}=\frac{a^2-bd}{3c}$.

3. $\frac{5}{4+b} : 2d-m :: \frac{b+c}{ad} : h(x-y)$.

4. $\frac{3abc}{m\sim n}=\frac{x}{y+8}-a(b-5)+2ch$.

5. $\frac{(a+d)(b\sim c)}{a+mn}+\frac{4c}{a+b}-\frac{h^4}{3a(b+c)}=\frac{b+c+d}{a-(x+y)}$.

47. At the close of an algebraic process, it is frequently necessary to restore the *numbers*, for which letters had been substituted, at the beginning. In doing this, the sign of multiplication must not be omitted, as it generally is, between factors, expressed by letters. Thus, if a stands for 3, and b for 4; the product ab is not 34, but 3×4, that is, 12. See Art. 29.

In the following examples,

Let $a=3$ And $d=6$
$b=4$ $m=8$
$c=2$ $n=10$.

Then, 1. $\frac{a+m}{cd}+\frac{bc-n}{3d}=\frac{3+8}{2\times6}+\frac{4\times2-10}{3\times6}$.

2. $\frac{m-(b+c)}{4ad}-\frac{3(c+n)}{2+d-b}+2(c-d)=$

3. $\frac{6n^2}{m-a}+5b(c+d-a)+\frac{7an-bm}{b^3-2ac}=$

48. An algebraic expression, in which numbers have been substituted for letters, may often be rendered much

more simple, by reducing several terms to one. This can not generally be done, while the letters remain. If $a+b$ is used for the sum of two quantities, a can not be united in the same term with b. But if a stands for 3, and b for 4, then $a+b=3+4=7$. The value of an expression, consisting of many terms may thus be found, by actually performing, with the numbers, the operations of addition, subtraction, multiplication, &c. indicated by the algebraic characters.

Find the value of the following expressions, in which the letters are supposed to stand for the same numbers, as in the preceding Article.

1. $\frac{ad}{c}+a+mn=\frac{3\times6}{2}+3+8\times10=9+3+80=92.$

2. $\frac{m+n}{b-a}+6(ab-d)+\frac{2ac}{n-m}=\frac{8+10}{4-3}+6(3\times4-6)+\frac{2\times3\times2}{10-8}=$

3. $2(a+b-c)+\frac{5d^2-(d-c)}{2(b-c)}-\frac{mn}{3m-2n}=$

4. $(a+b+c)\,(n-d)+\frac{m(d-a)}{3(b-c)}+\frac{6b+3n}{4d-5a}=$

5. $\frac{5b(2m+5a)}{3(d-c)\,(3a-m)}+\frac{7(ab-4c)}{4(11-2b)}=$

Positive and Negative Quantities.

49. To one who has just entered on the study of algebra, there is generally nothing more perplexing, than the use of what are called *negative* quantities. He supposes he is about to be introduced to a class of quantities which are entirely new; a sort of mathematical *nothings*, of which he can form no distinct conception. As positive quantities are *real*, he concludes that those which are negative must be *imaginary*. But this is owing to a misapprehension of the term negative, as used in the mathematics.

50. A NEGATIVE *quantity is one which is required to be* SUBTRACTED. When several quantities enter into a calculation, it is frequently necessary that some of them should be *added* together, while others are *subtracted*. The former are called affirmative or positive, and are marked with the sign +;

the latter are termed negative, and distinguished by the sign $-$. If, for instance, the profits of trade are the subject of calculation, and the *gain* is considered positive; the *loss* will be negative, because the latter must be *subtracted* from the former, to determine the clear profit. If the sums of a book account are brought into an algebraic process, the debt and the credit are distinguished by opposite signs. If a man on a journey is, by any accident, necessitated to return several miles, this backward motion is to be considered *negative*, because that, in determining his real progress, it must be subtracted from the distance which he has travelled in the opposite direction. If the *ascent* of a body from the earth be called positive, its *descent* will be negative. These are only different examples of the same general principle. In each of the instances, one of the quantities is to be *subtracted* from the other.

51. The terms positive and negative, as used in the mathematics, are merely *relative*. They imply that there is, either in the nature of the quantities, or in their circumstances, or in the purposes which they are to answer in calculation, some such *opposition* as requires that one should be *subtracted* from the other. But this opposition is not that of existence and non-existence, nor of one thing greater than nothing, and another less than nothing. For, in many cases, either of the signs may be, indifferently and at pleasure, applied to the very same quantity; that is, the two characters may change places. In determining the progress of a ship, for instance, her easting may be marked $+$, and her westing $-$; or the westing may be $+$, and the easting $-$. All that is necessary is, that the two signs be prefixed to the quantities, in such a manner as to show, which are to be added, and which subtracted. In different processes, they may be differently applied. On one occasion, a downward motion may be called positive, and on another occasion negative.

52. In every algebraic calculation, some one of the quantities must be fixed upon, to be considered positive. All other quantities which will *increase* this, must be positive also. But those which will tend to *diminish* it, must be negative. In a mercantile concern, if the *stock* is supposed to be positive, the *profits* will be positive; for they *increase* the stock; they are to be *added* to it. But the *losses* will be negative; for they *diminish* the stock; they are to be *subtracted* from it.

When a boat, in attempting to ascend a river, is occasionally driven back by the current; if the progress up the stream, to any particular point, is considered positive, every succeeding instance of *forward* motion will be positive, while the *backward* motion will be negative.

53. A negative quantity is frequently *greater*, than the positive one with which it is connected. But how, it may be asked, can the former be *subtracted* from the latter? The greater is certainly not *contained* in the less: how then can it be taken out of it? The answer to this is, that the greater may be supposed first to *exhaust* the less, and then to leave a remainder equal to the difference between the two. If a man has in his possession 1000 dollars, and has contracted a debt of 1500; the latter subtracted from the former, not only exhausts the whole of it, but leaves a balance of 500 against him. In common language, he is 500 dollars worse than nothing.

54. In this way, it frequently happens, in the course of an algebraic process, that a negative quantity is brought to *stand alone*. It has the sign of subtraction, without being connected with any other quantity, from which it is to be subtracted. This denotes that a previous subtraction has left a remainder, which is a part of the quantity subtracted. If the latitude of a ship which is 20 degrees north of the equator, is considered positive, and if she sails south 25 degrees; her motion first *diminishes* her latitude, then reduces it to *nothing*, and finally gives her 5 degrees of south latitude. The sign $-$ prefixed to the 25 degrees, is retained before the 5 to show that this is what remains of the *southward* motion, after balancing the 20 degrees of north latitude. If the motion southward is only 15 degrees, the remainder must be $+5$, instead of -5, to show that it is a part of the ship's *northern* latitude, which has been thus far diminished, but not reduced to nothing. The balance of a book account will be positive or negative, according as the debt or the credit is the greater of the two. To determine to which side the remainder belongs, the sign must be retained, though there is no other quantity, from which this is again to be subtracted, or to which it is to be added.

55. When a quantity continually decreasing is reduced to nothing, it is sometimes said to become afterwards *less than nothing*. But this is an exceptionable manner of

speaking.* No quantity can be really less than nothing. It may be diminished, till it vanishes, and gives place to an *opposite* quantity. The latitude of a ship crossing the equator, is first made less, then nothing, and afterwards *contrary* to what it was before. The north and south latitudes may therefore be properly distinguished, by the signs + and −; all the positive degrees being on one side of 0, and all the negative, on the other; thus,

$$+6, +5, +4, +3, +2, +1, 0, -1, -2, -3, -4, -5, \&c.$$

The numbers belonging to any other series of opposite quantities, may be arranged in a similar manner. So that 0 may be conceived to be a kind of *dividing point* between positive and negative numbers. On a thermometer, the degrees *above* 0 may be considered positive, and those *below* 0, negative.

56. A quantity is sometimes said to be *subtracted from* 0. By this is meant, that it belongs on the negative side of 0. But a quantity is said to be *added* to 0, when it belongs on the positive side. Thus, in speaking of the degrees of a thermometer, $0+6$ means 6 degrees *above* 0; and $0-6$, 6 degrees *below* 0.

Axioms.

57. The object of mathematical inquiry is, generally, to investigate some unknown quantity, and discover how *great* it is. This is effected, by comparing it with some other quantity or quantities already known. The dimensions of a stick of timber, are found, by applying to it a measuring rule of known length. The *weight* of a body is ascertained, by placing it in one scale of a balance, and observing how many pounds in the opposite scale, will equal it. And any quantity is determined, when it is found to be equal to some known quantity or quantities.

Let a and b be known quantities, and y, one which is unknown. Then y will become known, if it be discovered to be equal to the sum of a and b; that is, if

$$y=a+b.$$

* The expression "*less than nothing*," may not be wholly improper; if it is intended to be understood, not literally, but merely as a convenient phrase adopted for the sake of avoiding a tedious circumlocution; as we say "the sun rises," instead of saying "the earth rolls round, and brings the sun into view." The use of it in this manner, is warranted by Newton, Euler and others.

An expression like this, representing the equality between one quantity or set of quantities, and another, is called an *equation*. It will be seen hereafter, that much of the business of algebra consists in finding equations, in which some unknown quantity is shown to be equal to others which are known. But it is not often the fact, that the first comparison of the quantities, furnishes the equation required. It will generally be necessary to make a number of additions, subtractions, multiplications, &c. before the unknown quantity is discovered. But in all these changes, a constant equality must be preserved, between the two sets of quantities compared. This will be done, if, in making the alterations, we are guided by the following *axioms*. These are not inserted here, for the purpose of being proved; for they are self-evident. (Art. 7.) But as they must be continually introduced or implied, in demonstrations and the solutions of problems, they are placed together, for the convenience of reference.

58. *Axiom* 1. If the same quantity or equal quantities be *added* to equal quantities, their *sums* will be equal.

2. If the same quantity or equal quantities be *subtracted* from equal quantities, the *remainders* will be equal.

3. If equal quantities be *multiplied* into the same, or equal quantities, the *products* will be equal.

4. If equal quantities be *divided* by the same, or equal quantities, the *quotients* will be equal.

5. If the same quantity be both *added to* and *subtracted from* another, the value of the latter will not be altered.

6. If a quantity be both *multiplied* and *divided* by another, the value of the former will not be altered.

7. If to unequal quantities, equals be added, the greater will give the greater sum.

8. If from unequal quantities, equals be subtracted, the greater will give the greater remainder.

9. If unequal quantities be multiplied by equals, the greater will give the greater product.

10. If unequal quantities be divided by equals, the greater will give the greater quotient.

11. Quantities which are respectively equal to any other quantity are equal to each other.

12. The whole of a quantity is greater than a part.

This is, by no means, a *complete* list of the self-evident propositions, which are furnished by the mathematics. It is not necessary to enumerate them all. Those have been selected, to which we shall have the most frequent occasion to refer.

59. The investigations in algebra are carried on, principally, by means of a series of *equations* and *proportions*. But instead of entering directly upon these, it will be necessary to attend in the first place, to a number of processes, on which the management of equations and proportions depends. These preparatory operations are similar to the calculations under the common rules of arithmetic. We have addition, multiplication, division, involution, &c. in algebra, as well as in arithmetic. But this application of a common name, to operations in these two branches of the mathematics, is often the occasion of perplexity and mistake. The learner naturally expects to find addition in algebra the same as addition in arithmetic. They are in fact the same, in many respects: in *all* respects perhaps, in which the steps of the one will admit of a direct comparison, with those of the other. But addition in algebra is more *extensive*, than in arithmetic. The same observation may be made concerning several other operations in algebra. They are, in many points of view, the same as those which bear the same names in arithmetic. But they are frequently extended farther, and comprehend processes which are unknown to arithmetic. This is commonly owing to the introduction of *negative* quantities. The management of these requires steps which are unnecessary, where quantities of one class only are concerned. It will be important, therefore, as we pass along, to mark the *difference* as well as the *resemblance*, between arithmetic and algebra; and, in some instances, to give a new definition, accommodated to the latter.

SECTION II.

ADDITION.

Art. **60.** Addition *is the collecting of several algebraic expressions into one, so as to represent their aggregate value.*

This may always be done by the following

Rule.

Write the quantities to be added, one after another, with their signs; observing that a quantity to which *no* sign is prefixed, is to be considered positive.

61. Thus a added to b, is evidently, according to the algebraic notation, $a+b$. And a added to the sum of b and c, is $a+b+c$. And $a+b$, added to $c+d$, is $a+b+c+d$. In the same manner, if the sum of any quantities whatever, be added to the sum of any others, the expression for the whole, will contain all these quantities connected by the sign $+$.

62. Again, if the *difference* of a and b be added to c; the sum will be $a-b$ added to c, that is $a-b+c$. And if $a-b$ be added to $c-d$, the sum will be $a-b+c-d$. In one of the compound quantities added here, a is to be diminished by b, and in the other, c is to be diminished by d; the *sum* of a and c must therefore be diminished, both by b, and by d, that is, the expression for the sum total, must contain $-b$ and $-d$. On the same principle, all the quantities which, in the parts to be added, have the negative sign, must *retain* this sign in the amount. Thus $a+2b-c$, added to $d-h-m$, is $a+2b-c+d-h-m$.

63. The sign must be retained also, when a positive quantity is to be added, to a *single* negative quantity. If a be added to $-b$, the sum will be $-b+a$. Here it may be objected, that the negative sign prefixed to b, shows that it is to be *subtracted.* What propriety then can there be in *adding* the two quantities? In reply to this, it may be observed, that the sign prefixed to b while standing alone, signifies that

b is to be subtracted, *not from* a, but from some *other* quantity, which is not here expressed. Thus $-b$ may represent the *loss*, which is to be subtracted from the *stock* in trade. (Art. 50.) The object of the calculation, however, may not require that the value of this stock should be specified. But the loss is to be connected with a *profit* on some article. Suppose the profit is 2000 dollars, and the loss 400. The inquiry then is, what is the value of 2000 dollars profit, when connected with 400 dollars loss?

The answer is evidently $2000-400$, which shows that 2000 dollars are to be *added* to the stock, and 400 *subtracted* from it; or which will amount to the same, that the *difference* between 2000 and 400 is to be added to the stock.

These instances are sufficient to show that the preceding rule is adapted to every case in addition.

64. It is immaterial, in adding, in what *order* the terms are arranged. The sum of a and b and c is either $a+b+c$, or $a+c+b$, or $c+b+a$. For it evidently makes no difference, which of the quantities is added *first*. The sum of 6 and 3 and 9, is the same as 3 and 9 and 6, or 9 and 6 and 3.

And $a+m-n$, is the same as $a-n+m$. For it is plainly of no consequence, whether we first add m to a, and afterwards subtract n; or first subtract n, and then add m.

65. It is to be observed that addition in algebra does not always signify *augmentation*, as it does in arithmetic. The algebraic sum of two quantities which have opposite signs, may be less than either of them. Thus the aggregate value of 7 and -4, is not 11 but 3.

Reduction of Like Terms.

66. Though connecting quantities by their signs is all which is *essential* to addition; yet it is desirable to make the expression as simple as may be, by *reducing several terms to one*. The amount of $3a$, and $6b$, and $4a$, and $5b$, is

$$3a+6b+4a+5b.$$

But this may be abridged. The first and third terms may be brought into one; and so may the second and fourth. For 3 times a, and 4 times a, make 7 times a. And 6 times b, and 5 times b, make 11 times b. The sum when reduced is therefore $7a+11b$.

For making the reductions connected with addition, two rules are given, adapted to the two cases, in one of which, the quantities and signs are alike, and in the other, the quantities are alike, but the signs are unlike. Like quantities are the same *powers* of the same *letters*. (Art. 39.)

Case I.

67. To reduce several terms to one when the *quantities* are *alike*, and the *signs alike*,

Add the co-efficients, annex the common letter or letters, and prefix the common sign.

Thus to reduce $3b+7b$, that is $+3b+7b$ to one term, add the co-efficients 3 and 7; to the sum 10, annex the common letter b, and prefix the sign $+$. The expression will then be $+10b$. That 3 times any quantity, and 7 times the same quantity, make 10 times that quantity, needs no proof.

Examples.

$$\begin{array}{rrrrr}
bc & 2hx & 7b+\ \ xy & 4nxy+\ \ ar & cdxy+\ 3gm \\
2bc & hx & 8b+\ 3xy & 7nxy+3ar & 2cdxy+\ \ gm \\
9bc & 5hx & 2b+\ 2xy & 2nxy+5ar & 5cdxy+\ 7gm \\
3bc & 8hx & 6b+\ 5xy & nxy+\ \ ar & 7cdxy+\ 8gm \\
\hline
15bc & & 23b+11xy & & 15cdxy+19gm
\end{array}$$

The mode of proceeding will be the same, if the signs are *negative.*

Thus $-3bc-bc-5bc$, becomes, when reduced, $-9bc$.

And $-ax-3ax-2ax=-6ax$.

Or thus,

$$\begin{array}{rrrr}
-3b^2c & -7xy^3 & -\ 2ab-\ \ my & -\ b^2h-\ acnx \\
-\ b^2c & -\ xy^3 & -\ \ ab-\ 3my & -4b^2h-9acnx \\
-5b^2c & -3xy^3 & -\ 7ab-\ 8my & -\ b^2h-2acnx \\
\hline
-9b^2c & & -10ab-12my &
\end{array}$$

68. It may perhaps be asked here, as in Art. 63, what propriety there is, in *adding* quantities, to which the negative

sign is prefixed; a sign which denotes *subtraction?* The answer to this is, that when the negative sign is applied to several quantities, it is intended to indicate that these quantities are to be subtracted, *not from one another*, but from some *other* quantity marked with the contrary sign. Suppose that, in estimating a man's property, the sum of money in his possession is marked +, and the debts which he owes are marked −. If these debts are 200, 300, 500 and 700 dollars, and if a is put for 100; they will together be $-2a-3a-5a-7a$. And the several terms reduced to one, will evidently be $-17a$, that is, 1700 dollars.

69. Before attending to the second case, it is to be observed that *two* terms may be reduced to one, when the *quantities* are *alike*, but the *signs unlike*, by the following rule.

Take the less co-efficient from the greater, to the difference annex the common letter or letters, and prefix the sign of the greater co-efficient.

Thus, instead of $8a-6a$, we may write $2a$.

And instead of $7b-2b$, we may put $5b$.

For the simple expression, in each of these instances, is equivalent to the compound one for which it is substituted.

To	$+6b$	$-8a^2$	$5bc$	$-n^2x$	$-dy+6m$	$-a-2b^2y$
Add	$-4b$	$+5a^2$	$-7bc$	$8n^2x$	$4dy-m$	$7a-b^2y$
Sum	$+2b$		$-2bc$		$3dy+5m$	

70. Here again, it may excite surprise, that what appears to be subtraction, should be introduced under addition. But this subtraction is strictly speaking, no part of the addition. It belongs to a consequent *reduction*. Suppose $6b$ is to be added to $a-4b$. The sum is $a-4b+6b$. (Art. 60.) But this expression may be rendered more simple. As it now stands, $4b$ is to be subtracted from a, and $6b$ added. But the amount will be the same, if, without subtracting any thing, we add $2b$, making the whole $a+2b$. And in all similar instances, the *balance* of two quantities, may be substituted for the quantities themselves.

If two *equal* quantities have *contrary signs*, they destroy each other, and may be cancelled. Thus $+6b-6b=0$: And $3\times6-18=0$: And $7bc-7bc=0$.

Case II.

71. To reduce several terms to one, when the *quantities* are *alike*, but have *different signs*,

Find the difference between the sum of the positive and the sum of the negative co-efficients; to this prefix the sign of the greater sum, and annex the common letter or letters.

Ex. 1. Reduce $13b+6b+b-4b-5b-7b$, to one term.

$$\left.\begin{array}{ll} \text{By Art. 67,} & 13b+6b+\ b=\ \ 20b \\ \text{And} & -4b-5b-7b=-16b \end{array}\right\}$$

By Art. 69, $20b-16b=\ 4b$, which is the value of all the given quantities, taken together.

Ex. 2. Reduce $3xy-xy+2xy-7xy+4xy-9xy-6xy+\ 7xy$.

The positive terms are	$3xy$	The negative terms are	$-\ \ xy$
	$2xy$		$-\ 7xy$
	$4xy$		$-\ 9xy$
	$7xy$		$-\ 6xy$
And their sum is	$16xy$		$-23xy$

Then $16xy-23xy=-7xy$.

Ex. 3. $c^2x-5c^2x+2c^2x-7c^2x-c^2x+9c^2x+4c^2x-3c^2x=0$.

4. $5h^3xy-2h^3xy+h^3xy+9h^3xy-4h^3xy=$

5. $-abch-6abch+7abch-8abch+2abch=$

72. If the *letters*, in the several terms to be added, are different, they can only be placed after each other, with their proper signs. They can not be united in one simple term. If $4b$, and $-6y$, and $3x$, and $17h$, and $-5d$, and 6, be added, their sum will be

$$4b-6y+3x+17h-5d+6. \quad \text{(Art. 60.)}$$

Different letters can no more be united in the same term, than dollars and guineas can be added, so as to make a single sum. Six guineas and four dollars are neither ten guineas nor ten dollars. Seven hundred and five dozen, are neither 12 hundred nor 12 dozen. But in such cases, the algebraic

signs serve to show how the different quantities stand related to each other; and to indicate future operations, which are to be performed, whenever the letters are converted into numbers. In the expression $a+6$, the two terms can not be united in one. But if a stands for 15, and if, in the course of a calculation, this number is restored; then $a+6$ will become $15+6$, which is equivalent to the single term 21. In the same manner, $a-6$, becomes $15-6$, which is equal to 9. The signs keep in view the relations of the quantities, till an opportunity occurs of reducing several terms to one.

73. When the quantities to be added contain several terms which are *alike*, and several which are *unlike*, it will be convenient to arrange them in such a manner, that the similar terms may stand one under another.

To	$3bc-6d+2b-3y$	These may be arranged thus:
Add	$-3bc+x-3d+bg$	$3bc-6d+2b-3y+\ x+bg+b$
And	$2d+y+3x+b$	$-3bc-3d\qquad +\ y+3x$
		$2d$

The sum will be $\quad -7d+2b-2y+4x+bg+b$

Examples.

1. Add $7-xy$, to $ab+2$, and $3xy-8h-6$.

   The sum, when reduced, is $\quad 3+2xy+ab-8h$.

2. Add $2y^2-5x-2ay$, to $bn-6-2y^2+1$.

   Ans. $bn-5-5x-2ay$.

3. Add $bc+2x^2-axy$, to $9-x^2-h$, and $6h-7-4x^2$.
4. Add $5y^3-9+ah+1$, to $12ah-y^3-10$.
5. Add $bx^2+4-y-3a^3x$, to $9bx^2-13+y+3a^3x$.
6. Add $5hn^2+3ab-hn^2+x$, to $hn^2-x-7ab$.
7. Add $4x^2-3bx-y$, to $x^2-ay+bx-5n$.
8. Add $ah+2mn-3x$, to $4an-2ah+4x$.

SECTION III.

SUBTRACTION.

ART. 74. SUBTRACTION *is finding the* DIFFERENCE *of two quantities, or sets of quantities.*

Particular rules might be given, for the several cases in subtraction. But it is more convenient to have one general rule, founded on the principle, that *taking away a positive* quantity from an algebraic expression, is the same in effect, as *annexing* an equal *negative* quantity; and taking away a negative quantity is the same, as annexing an equal positive one.

Suppose $+b$ is to be subtracted from	$a+b$
Taking away $+b$, from $a+b$, leaves	a
And annexing $-b$, to $a+b$, gives	$a+b-b$
But by axiom 5th, $a+b-b$ is equal to	a

That is, *taking away* a *positive* term, from an algebraic expression, is the same in effect, as *annexing* an equal *negative* term.

Again, suppose $-b$ is to be subtracted from	$a-b$
Taking away $-b$, from $a-b$, leaves	a
And annexing $+b$, to $a-b$, gives	$a-b+b$
But $a-b+b$ is equal to	a

That is, *taking away* a *negative* term, is equivalent to *annexing* a *positive* one.

If an estate is encumbered with a debt; to cancel this debt is to add so much to the value of the estate. Subtracting an item from one side of a book account, will produce the same alteration in the balance, as adding an equal sum to the opposite side.

So, if there are *several* terms in the subtrahend, each of them, instead of being *taken from* the minuend, may be *annexed*, with its sign changed.

To place this in another point of view. Suppose that $c-d$ is to be subtracted from $a-b$. If we first subtract c, the

remainder will be $a-b-c$, which is too small by d, inasmuch as we were to subtract, not the whole of c, but c diminished by d. Adding d therefore, we obtain $a-b-c+d$ for the required difference. The explanation will be the same, if there are more than two terms in the subtrahend or minuend. Hence, to perform subtraction in algebra, we have the following general

Rule.

75. *Change the signs of all the quantities to be subtracted, or suppose them to be changed, from + to −, or from − to +, and then proceed as in addition.*

This rule may be deduced from the following considerations, as well as from those already stated.

If two quantities be increased or diminished equally, their difference will remain unaltered. Suppose it is required to find the difference between $a+b$, and $c-d-h$. Before seeking the difference, let us add to each quantity, the expression $-c+d+h$, which consists of the terms of the subtrahend, with their signs changed. The minuend will become $a+b-c+d+h$. And the subtrahend will become $c-d-h-c+d+h$. This reduced becomes nothing, since c is destroyed by $-c$, $-d$ by $+d$, and $-h$ by $+h$. (Art. 70.) Now as the subtrahend is nothing, the remainder must be the same as the minuend, namely, $a+b-c+d+h$. Thus it appears that the required difference is obtained, by adding to the original minuend, all the terms of the subtrahend with their signs changed. And hence we derive the general rule for subtraction, as stated above.

Although this rule is adapted to every case of subtraction; yet there may be an advantage in giving some of the examples in distinct classes.

76. In the first place, the signs may be *alike*, and the minuend *greater* than the subtrahend.

From	$+28$	$16b$	$14da$	-28	$-16b$	$-14da$
Subtract	$+16$	$12b$	$6da$	-16	$-12b$	$-6da$
Difference	$+12$	$4b$	$8da$	-12	$-4b$	$-8da$

Here, in the first example, the + before 16 is supposed to be changed into −, and then, the signs being unlike, the two

terms are brought into one, by reduction, as in addition. (Art. 69.) The two next examples are subtracted in the same way. In the three last, the − in the subtrahend, is supposed to be changed into +.

It may be well for the learner, at first, to write out the examples; and actually to change the signs, instead of merely conceiving them to be changed. But when he has become familiar with the operation, the latter method is much to be preferred.

This case is the same as subtraction in *arithmetic*. The two next cases do not occur in common arithmetic.

77. In the second place, the signs may be alike, and the minuend *less* than the subtrahend.

From	$+16$	$12b$	$6da$	-16	$-12b$	$-6da$
Subtract	$+28$	$16b$	$14da$	-28	$-16b$	$-14da$
Difference	-12	$-4b$	$-8da$	$+12$	$4b$	$8da$

The same quantities are given here, as in the preceding article, for the purpose of comparing them together. But the minuend and subtrahend are made to change places. The mode of subtracting is the same. In this class, a *greater* quantity is taken from a *less*; in the preceding, a *less* from a *greater*. By comparing them, it will be seen, that there is no difference in the answers, except that the *signs* are *opposite*. Thus $16b-12b$ is the same as $12b-16b$, except that one is $+4b$, and the other $-4b$. That is, a greater quantity subtracted from a less, gives the same result, as the less subtracted from the greater, except that the one is positive, and the other negative. See Art. 53 and 54.

78. In the third place, the *signs* may be *unlike*.

From	$+28$	$+16b$	$+14da$	-28	$-16b$	$-14da$
Sub.	-16	$-12b$	$-6da$	$+16$	$+12b$	$+6da$
Dif.	$+44$	$28b$	$20da$	-44	$-28b$	$-20da$

From these examples, it will be seen that subtraction in algebra does not always imply *diminution*, as it does in arithmetic. The *difference* between a positive and a negative quantity, is *greater* than either of the two quantities. In a thermometer, the difference between 28 degrees above cipher, and 16 below, is 44 degrees. The difference between gain-

ing 1000 dollars in trade, and losing 500, is equivalent to 1500 dollars.

79. Subtraction may be *proved,* as in arithmetic, by adding the remainder to the subtrahend. The sum ought to be equal to the minuend, upon the obvious principle, that the difference of two quantities added to one of them, is equal to the other. This serves not only to correct any particular error, but to verify the general rule.

From	$2xy-1$	$3a^2+xy$	$hy-ah$	my^2-ny
Sub.	$-xy+7$	$5a^2+xy$	$5hy-6ah$	$7my^2-5ny$
Dif.	$3xy-8$		$-4hy+5ah$	

From	$3abm-xy$	$-1-3xy^3$	$ax+7b$	$-2n^3+3bh$
Sub.	$-7abm+6xy$	$-15+2xy^3$	$-4ax+15b$	$+n^3+3bh$
Rem.	$10abm-7xy$		$5ax-8b$	

80. When there are *several terms alike,* they may be reduced as in addition.

1. From ab, subtract $3am+am+7am+2am+6am$.

Ans. $ab-3am-am-7am-2am-6am=ab-19am$. (Art. 67.)

2. From y, subtract $-a-a-a-a$.

Ans. $y+a+a+a+a=y+4a$.

3. From $ax-bc+3ax+7bc$, subtract $4bc-2ax+bc+4ax$.

Ans. $ax-bc+3ax+7bc-4bc+2ax-bc-4ax$
$=2ax+bc$. (Art. 71.)

4. From $3x^2-2ay^3+bc$, subtract $x^2-4bc+ay^3-3x^2$.

81. When the *letters* in the minuend are *different* from those in the subtrahend, the latter are subtracted, by first changing the signs, and then placing the several terms one after another, as in addition. (Art. 72.)

From $3ab+8-my+dh$, subtract $x-dr+4hy-bmx$.

Ans. $3ab+8-my+dh-x+dr-4hy+bmx$.

82. The sign $-$, placed before the marks of *parenthesis,* which include a number of quantities, indicates that the included polynomial is to be subtracted, and therefore requires, that when those marks are removed, the signs of all the terms of the polynomial should be changed.

Thus $a-(b-c+d)$ signifies that the quantities b, $-c$ and $+d$, are to be subtracted from a. The expression will then become $a-b+c-d$.

2. $13ad+xy+d-(7ad-xy+d+hm-ry)=6ad+2xy-hm+ry$.

3. $7abc-8+7x-(3abc-8-dx+r)=4abc+7x+dx-r$.

4. $5x-y^2+hn-(2hn-4y^2-x+ah+6x-ny)=$

5. $2x^3+ay-12-(1-x^3+2h-13-3ay-c)=$

6. $ab-3ch-2-(3-d-5ab-ch+y)=$

83. On the other hand, when a number of quantities are introduced within the marks of parenthesis, with $-$ immediately preceding; the signs must be changed.

Thus $-m+b-dx+3h=-(m-b+dx-3h)$.

Polynomials may accordingly be written in various ways. For example, $x-ab+9-y^2-7h$, may be changed to $x-(ab-9+y^2+7h)$, or $x-ab-(-9+y^2+7h)$, or $x-ab+9-(y^2+7h)$, or $x-(ab-9)-(y^2+7h)$; each of these expressions being equivalent to the first.

SECTION IV.

MULTIPLICATION.

ART. **84.** IN addition, one quantity is connected with another. It is frequently the case, that the quantities brought together are *equal;* that is, a quantity is added to *itself.*

As $a+a=2a$ $\qquad a+a+a+a=4a$

$a+a+a=3a$ $\qquad a+a+a+a+a=5a$, &c.

This repeated addition of a quantity to itself, is what was originally called *multiplication.* But the term, as it is now used, has a more extensive signification. We have frequent

occasion to repeat, not only the *whole* of a quantity, but a certain *portion* of it. If the stock of an incorporated company is divided into shares, one man may own ten of them, another five, and another a *part* only of a share, say two-fifths. When a dividend is made, of a certain sum on a share, the first is entitled to *ten* times this sum, the second to *five* times, and the third to only *two-fifths* of it. As the apportioning of the dividend, in each of these instances, is upon the same principle, it is called multiplication in the last, as well as in the two first.

According to this view of the subject;

85. *Multiplying by a* WHOLE NUMBER *is taking the multiplicand as many times, as there are units in the multiplier.*

Multiplying by 1, is taking the multiplicand *once*, as a.

Multiplying by 2, is taking the multiplicand *twice*, as $a+a$.

Multiplying by 3, is taking the multiplicand *three times*, as $a+a+a$, &c.

Multiplying by a FRACTION *is taking a certain* PORTION *of the multiplicand as many times, as there are like portions of a unit in the multiplier.**

Multiplying by $\frac{1}{3}$, is taking $\frac{1}{3}$ of the multiplicand, *once*, as $\frac{1}{3}a$.

Multiplying by $\frac{2}{3}$, is taking $\frac{1}{3}$ of the multiplicand, *twice*, as $\frac{1}{3}a+\frac{1}{3}a$.

Multiplying by $\frac{3}{3}$, is taking $\frac{1}{3}$ of the multiplicand, *three times*.

Hence, if the multiplier is *a unit*, the product is *equal* to the multiplicand: If the multiplier is *greater* than a unit, the product is greater than the multiplicand: And if the multiplier is *less* than a unit, the product is less than the multiplicand.

86. Every multiplier is to be considered a *number*. We sometimes speak of multiplying by a given *weight* or *measure*, a sum of *money*, &c. But this is abbreviated language. If construed literally, it is absurd. Multiplying is taking either the whole or a part of a quantity, a certain *number of times*. To say that one quantity is repeated as many times, as another is *heavy*, is nonsense. But if a part of the weight of a body be fixed upon as a *unit*, a quantity may be multi-

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* See Note B.

plied by a *number* equal to the number of these parts contained in the body. If a diamond is sold by weight, a particular price may be agreed upon for each *grain*. A grain is here *the unit;* and it is evident that the value of the diamond, is equal to the given price repeated as many times, as there are grains in the whole weight. We say concisely, that the price is multiplied by the *weight;* meaning that it is multiplied by a *number* equal to the number of grains in the weight. In a similar manner, any quantity whatever may be supposed to be made up of parts, each being considered a *unit*, and any number of these may become a multiplier.

87. As multiplying is taking the whole or a part of a quantity a certain number of times, it is evident that the *product*, must be of the same nature as the *multiplicand*.

If the multiplicand is an abstract *number*, the product will be a number.

If the multiplicand is *weight*, the product will be weight. If the multiplicand is a *line*, the product will be a line. *Repeating* a quantity does not alter its nature. It is frequently said, that the product of two lines is a *surface*, and that the product of three lines is a *solid*. But these are abbreviated expressions, which if interpreted literally are not correct.

88. The multiplication of *fractions* will be the subject of a future section. We have first to attend to multiplication by *positive whole numbers*. And there are here two cases to be considered; one, in which the factors are monomials; and the other, in which they are polynomials.

Multiplication of Monomials.

89. Multiplying by a whole number, as above defined (Art. 85.) is taking the multiplicand as many times, as there are units in the multiplier. Suppose a is to be multiplied by b, and that b stands for 3. There are then, three units in the multiplier b. The multiplicand must therefore be taken three times; thus, $a+a+a=3a$, or ba. So that,

Multiplying two letters together is nothing more, than writing them one after the other, either with, or without the sign of multiplication between them.

Thus b multiplied into c is $b\times c$, or bc. And x into y, is $x\times y$, or $x.y$, or xy. See Art. 20.

90. If *more than two* letters are to be multiplied, they must be connected in the same manner.

Thus a into b and c, is cba. For by the last article, a into b, is ba. This product is now to be multiplied into c. If c stands for 5, then ba is to be taken five times thus,

$$ba+ba+ba+ba+ba=5ba, \text{ or } cba.$$

The same explanation may be applied to any number of letters. Thus, am into xy, is $amxy$. And bh into mrx, is $bhmrx$.

91. It is immaterial *in what order* the letters are arranged. The product ba is the same as ab. Three times five is equal to five times three. Let the number 5 be represented by as many points, in a *horizontal* line; and the number 3, by as many points in a *perpendicular* line.

$$\begin{array}{ccccc} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{array}$$

Here it is evident that the *whole* number of points is equal either to the number in the *horizontal* row *three* times repeated, or to the number in the *perpendicular* row *five* times repeated; that is, to 5×3, or 3×5. This explanation may be extended to a series of factors consisting of any numbers whatever. For the product of two of the factors may be considered as one number. This may be placed before or after a third factor; the product of three, before or after a fourth; &c.

Thus $24=4\times6$ or $6\times4=4\times3\times2$ or $4\times2\times3$ or $2\times3\times4$

The product of a, b, c and d, is $abcd$, or $acdb$, or $dcba$, or $badc$.

It will generally be convenient, however, to place the letters in *alphabetical* order.

92. Powers *of the same quantity are multiplied by adding their exponents.*

Thus the product of a^2 into a^3 is a^5. For aa is the same as a^2, and aaa the same as a^3; and the product of aa into aaa is $aaaaa$, that is, a^5. Also, $b^3\times b^4=b^7$, and $x^2\times x^{10}=x^{12}$.

It is to be observed, that when a letter has no exponent, *one* is to be understood. Thus $h\times h^3$, is the same as $h^1\times h^3$, that is, h^4.

93. *When the letters have numerical* CO-EFFICIENTS, *these must be multiplied together, and prefixed to the product of the letters.*

Thus, $3a$ into $2b$, is $6ab$. For if a into b is ab, then 3 times a into b, is evidently $3ab$; and if, instead of multiplying by b, we multiply by *twice* b, the product must be *twice* as great; that is $2\times3ab$ or $6ab$.

Multiply	$9ab$	$5cd$	$3dy$	$7amh$	$7bdh$	$h\ x^3$
Into	$3xy$	$12bh$	my^2	$9by$	x	$13h^2x^2$
Product	$27abxy$		$3dmy^3$		$7bdhx$	

If either of the factors consists of figures *only*, these must be multiplied into the co-efficients and letters of the other factors.

Thus $3ab$ into 4, is $12ab$. And 36 into $2x$, is $72x$. And 24 into hy, is $24hy$.

94. The numeral co-efficients of *several* fellow-factors may be brought together by multiplication.

Thus $2a\times3b$ into $4a\times5b$ is $2a\times3b\times4a\times5b$, or $120a^2b^2$.

For the co-efficients are *factors* (Art. 31.), and it is immaterial in what *order* these are arranged. (Art. 91.) So that

$2a\times3b\times4a\times5b=2\times3\times4\times5\times a\times a\times b\times b=120a^2b^2$.

The product of $3a\times4bh$ into $5m\times6y$, is $360abhmy$.

The product of $4b\times6d$ into x, is $24bdx$.

MULTIPLICATION OF POLYNOMIALS.

95. *If the* MULTIPLICAND *is a* COMPOUND *quantity, each of its terms must be multiplied into the multiplier.*

Thus $b+c+d$ into a, is $ab+ac+ad$. For the whole of the multiplicand is to be taken as many times, as there are units in the multiplier. If then a, stands for 3, the repetitions of the multiplicand are,

$$\begin{array}{c} b+c+d \\ b+c+d \\ b+c+d \\ \hline \end{array}$$

And their sum is $3b+3c+3d$, that is, $ab+ac+ad$.

Multiply	$d+2xy$	h^2+2n	$3hl+1$	$ax+2hy+5$
Into	$3b$	$5hx$	my	$3y$
Product	$3bd+6bxy$		$3hlmy+my$	

96. The preceding instances must not be confounded with those in which several *factors* are connected by the sign $\times$, or by a point. In the latter case, the multiplier is to be written before the other factors *without being repeated.*

The product of $b\times d$ into a, is $ab\times d$, and not $ab\times ad$. For $b\times d$ is bd, and this into a, is abd. (Art. 90.) The expression $b\times d$ is not to be considered, like $b+d$, a *compound quantity* consisting of two terms. Different terms are always separated by + or −. (Art. 38.) The product of $b\times h\times m\times y$ into a, is $a\times b\times h\times m\times y$ or $abhmy$. But $b+h+m+y$ into a, is $ab+ah+am+ay$. The product of 3×5 into 4, is 15×4 or 60, and not $4\times 3\times 4\times 5$, which is 240. But the product of $3+5$ into $4=4\times 3+4\times 5=12+20=32$.

97. *If* BOTH *the* FACTORS *are* COMPOUND *quantities, each term in the multiplier must be multiplied into each in the multiplicand.*

Thus $a+b$ into $c+d$ is $ac+ad+bc+bd$.

For the units in the multiplier $a+b$ are equal to the units in a *added* to the units in b. Therefore the product produced by a, must be added to the product produced by b.

$$\left.\begin{array}{l}\text{The product of } c+d \text{ into } a \text{ is } ac+ad\\ \text{The product of } c+d \text{ into } b \text{ is } bc+bd\end{array}\right\}\quad \text{(Art. 95.)}$$

The product of $c+d$ into $a+b$ is therefore $ac+ad+bc+bd$.

Mult.	$3x+d$	$3bx+a$	$a+1$
Into	$2a+hm$	$2b^2+3h$	$3x+4$
Prod.	$6ax+2ad+3hmx+dhm$		$3ax+3x+4a+4$

4. Mult. $2h+7$ into $6d+1$. Prod. $12dh+42d+2h+7$.
5. Mult. $ax+b^2+c$ into $n+3+7a^2$. Prod.
6. Mult. $1+x+2hy$ into $5a+3b+6$. Prod.

98. When two or more terms in the product are *alike*, it will be expedient to *set one under the other*, and then to unite them, by the rules for reduction in addition.

Mult.	$b+2a$	$b+c+2$	$x+6+a^2$
Into	$b+2a$	$b+c+3$	$2y+3b+1$
	b^2+2ab	$b^2+bc+2b$	
	$+2ab+4a^2$	$bc \quad +c^2+2c$	
		$+3b \quad +3c+6$	
Prod.	$b^2+4ab+4a^2$	$b^2+2bc+5b+c^2+5c+6$	

4. Mult. $h+a+1$ into $3h+2a+5$. Prod.

5. Mult. $2a+3+xy$ into $a+2+3xy$. Prod.

6. Mult. $c+3d+2e$ into $b\times2d\times x$. Prod.

99. The examples in multiplication thus far have been confined to positive quantities. But in algebra, multiplication is performed also with negative quantities. And it will now be necessary to consider in what manner the result will be affected, by multiplying *positive* and *negative* quantities together. We shall find,

That + into + produces +
− into + −
+ into − −
− into − +

All these may be comprised in one *general rule*, which it will be important to have always familiar.

If the SIGNS *of the factors are* ALIKE, *the sign of the product will be positive; but if the signs of the factors are* UNLIKE, *the sign of the product will be negative.*

100. The *first* case, that of + into +, needs no farther illustration.

The *second* is − into +, that is, the multiplicand is negative, and the multiplier positive. Here $-a$ into $+4$ is $-4a$. For the repetitions of the multiplicand are, $-a-a-a-a=-4a$. And $-a$ into $+b$ is $-a$ repeated b times, that is, $-ba$.

Mult.	$b-3a$	$5x-a$	$h-3d-4$	$y-a-3-2h$
Into	$6y$	$b+2y$	$2y$	$2x+3b$
Prod.	$6by-18ay$		$2hy-6dy-8y$	

101. The *third* case is that in which the multiplicand is positive, but the multiplier negative.

The effect of multiplying by a negative quantity is best shown by the aid of a compound multiplier. Thus, it is easy to ascertain the product of a into -4, from the product of a into the binomial $6-4$. As $6-4$ is equal to 2, the product will be equal to $2a$. This is *less* than the product of 6 into a. To obtain then the product of the compound multiplier $6-4$ into a, we must *subtract* the product of the negative part, from that of the positive part.

$$\left.\begin{array}{lr}\text{Multiplying} & a\\ \text{Into} & 6-4\end{array}\right\}\text{ is the same as }\left\{\begin{array}{lr}\text{Multiplying} & a\\ \text{Into} & 2\end{array}\right.$$

And the product $6a-4a$, is the same as the product $2a$.

Therefore a into -4, is $-4a$. And this must be the product, as well when -4 stands alone, as when it forms a part of a compound multiplier. For every negative quantity must be supposed to have a reference to some other which is positive; though the two may not always stand in connection, when the multiplication is to be performed.

If the multiplier had been $6+4$ instead of $6-4$, the two products $6a$ and $4a$ must have been *added.*

$$\left.\begin{array}{lr}\text{Multiplying} & a\\ \text{Into} & 6+4\end{array}\right\}\text{ is the same as }\left\{\begin{array}{lr}\text{Multiplying} & a\\ \text{Into} & 10\end{array}\right.$$

And the prod. $6a+4a$ is the same as the product $10a$.

This shows at once the difference between multiplying by a *positive* factor, and multiplying by a *negative* one. In the former case, the sum of the repetitions of the multiplicand is to be *added to* other quantities, and in the latter, *subtracted from* them.

Mult.	$a+b$	$bx+3+2y^2$	$3h+3$
Into	$b-x$	$am-bc$	$ad-6$
Prod.	$ab+b^2-ax-bx$		$3adh+3ad-18h-18$

102. If *two negatives* be multiplied together, the product will be positive; $-4\times-a=+4a$. In this case, as in the preceding, the repetitions of the multiplicand are to be *subtracted*, because the multiplier has the negative sign. These repetitions, if the multiplicand is $-a$, and the multiplier -4, are $-a-a-a-a=-4a$. But this is to be subtracted by changing the sign. It then becomes $+4a$.

Suppose $-a$ is multiplied into $(6-4)$. As $6-4=2$, the product is, evidently, *twice* the multiplicand, that is $-2a$. But if we multiply $-a$ into 6 and 4 separately; $-a$ into 6 is $-6a$, and $-a$ into 4 is $-4a$. (Art. 100.) As in the multiplier, 4 is to be subtracted from 6; so, in the product, $-4a$ must be subtracted from $-6a$. Now $-4a$ becomes by subtraction $+4a$. The whole product then is $-6a+4a$, which is equal to $-2a$. Or thus,

$$\left.\begin{matrix}\text{Multiplying} & -a\\ \text{Into} & 6-4\end{matrix}\right\}\text{ is the same as }\left\{\begin{matrix}\text{Multiplying} & -a\\ \text{Into} & 2\end{matrix}\right.$$

And the prod. $-6a+4a$, is equal to the product $-2a$.

It is often considered a great mystery, that the product of two negatives should be positive. But it amounts to nothing more than this, that the subtraction of a negative quantity, is equivalent to the addition of a positive one; (Art. 74.) and therefore, that the *repeated* subtraction of a negative quantity, is equivalent to a *repeated* addition of a positive one. Taking off from a man's hands a debt of ten dollars every month, is adding ten dollars a month to the value of his property.

Mult.	$a-4$	$a-2bx+h^2$	$3ay-b$
Into	$3b-6$	$3c-5$	$6x-1$
Prod.	$3ab-12b-6a+24$		$18axy-6bx-3ay+b$

4. Multiply $b^2-5-2cx$ into $7-b-cy$.

5. Multiply $3ax+h-2$ into $5y^2-n+1$.

103. As a negative multiplier changes the sign of the quantity which it multiplies; if there are *several* negative factors to be multiplied together,

The *two first* will make the product *positive;*
The *third* will make it *negative;*
The *fourth* will make it *positive;* &c.

$$\left.\begin{aligned} \text{Thus } -a\times -b &= +ab \\ +ab\times -c &= -abc \\ -abc\times -d &= +abcd \\ +abcd\times -e &= -abcde \end{aligned}\right\}\text{the product of}\begin{cases} \textit{two factors.} \\ \textit{three.} \\ \textit{four.} \\ \textit{five.} \end{cases}$$

That is, the product of any *even* number of negative factors is *positive;* but the product of any *odd* number of negative factors is *negative.*

Thus $-a\times -a=a^2$ And $-a\times -a\times -a\times -a=a^4$

$-a\times -a\times -a=-a^3$ $-a\times -a\times -a\times -a\times -a=-a^5$

The product of several factors which are all *positive,* is invariably positive.

104. Positive and negative terms may frequently *balance* each other, so as to disappear in the product. (Art. 70.) A star is sometimes put in the place of a deficient term.

Mult.	$2a\ -b$	$x+a^2$	a^2+ab+b^2
Into	$2a\ +b$	$x-a^2$	$a\ -b$
	$4a^2-2ab$		$a^3+a^2b+ab^2$
	$+2ab-b^2$		$-a^2b-ab^2-b^3$
Prod.	$4a^2\ \ *\ \ -b^2$		$a^3\ \ *\ \ *\ \ -b^3$

105. For many purposes, it is sufficient merely to *indicate* the multiplication of compound quantities, without actually multiplying the several terms. Thus the product of

$a+b+c$ into $h+m+y$, is $(a+b+c)\times(h+m+y)$, or simply $(a+b+c)\ (h+m+y)$.

The product of

$a+m$ into $h+x$ and $d+y$, is $(a+m)\ (h+x)\ (d+y)$.

When the several terms are multiplied in form, the expression is said to be *expanded.* Thus,

$(a+b)\ (c+d)$ becomes when expanded $ac+ad+bc+bd$.

106. With a given multiplicand, the less the multiplier, the less will be the product. If then the multiplier be reduced to *nothing*, the *product* will be nothing. Thus $a\times 0=0$. And if 0 be one of *any number* of fellow-factors, the product of the whole will be nothing.

Thus, $ab\times c\times 3d\times 0=3abcd\times 0=0.$

And $(a+b)\,(c+d)\,(h-m)\times 0=0.$

107. Although, for the sake of illustrating the different points in multiplication, the subject has been drawn out into a considerable number of particulars; yet it will scarcely be necessary for the learner, after he has become familiar with the examples, to burden his memory with any thing more than the following *general rule*.

Multiply the letters and co-efficients of each term in the multiplicand, into the letters and co-efficients of each term in the multiplier; and prefix to each term of the product, the sign required by the principle, that like signs produce +, and different signs —.

When *like* terms occur in the product, they are to be united. See Art. 71 and 98.

1. Mult. $3a-x+2$ into $a-4x-1$.
2. Mult. $xy\times 3a\times 5$ into $2n+1-3h^2$.
3. Mult. $5(x-3ab)$ into $y\times 2\times 5h\times 3$.
4. Mult. $3(2a^2+bc-1)$ into $d(6-2x-1)$.
5. Mult. $ax+5x-h-2$ into $(c+d)(x-h)$.
6. Mult. $5x^2y-(b-3c)$ into $(b-1)\,(n+1)$.
7. Mult. $ay-7+b(c-m)$ into $-(2n-1+3x)$.

Multiplication by Detached Co-efficients.

108. There are certain cases, in which the numeral co-efficients may be employed, apart from the letters, in obtaining the product of two polynomials.

Suppose that $a^2+3ax+2x^2$ is to be multiplied by $2a^2+5ax+x^2$. The operation performed in the usual way, is as follows.

$$\begin{array}{l}
a^2+\ 3a\ x+\ 2x^2 \\
2a^2+\ 5a\ x+\ \ x^2 \\
\hline
2a^4+\ 6a^3x+\ 4a^2x^2 \\
\quad\ +\ 5a^3x+15a^2x^2+10ax^3 \\
\qquad\qquad\quad +\ \ a^2x^2+\ 3ax^3+2x^4 \\
\hline
2a^4+11a^3x+20a^2x^2+13ax^3+2x^4
\end{array}$$

Now with regard to the letters and exponents, it is easy to see, before multiplying, that as each term in the multiplier and each in the multiplicand is of the second degree, each term in the product will be of the fourth degree: and it is obvious that the first term will contain a^4; the last, x^4; and the intervening terms, a^3x, a^2x^2, ax^3.

We may then, in multiplying, proceed with the co-efficients alone, as if they were accompanied with letters, and after having thus obtained the co-efficients of the product, annex to them their proper letters and exponents.

This process is exhibited below.

$$\begin{array}{ll}
\text{Co-efficients of } a^2+3ax+2x^2, & 1+\ 3+2 \\
\text{Co-efficients of } 2a^2+5ax+\ x^2, & 2+\ 5+1 \\
\cline{2-2}
 & 2+\ 6+4 \\
 & \quad\ 5+15+10 \\
 & \qquad\quad\ 1+\ 3+2 \\
\cline{2-2}
\text{Co-efficients of the product,} & 2+11+20+13+2 \\
\hline
\text{The product itself,} & 2a^4+11a^3x+20a^2x^2+13ax^3+2x^4
\end{array}$$

The same method may be pursued, when any of the co-efficients are negative. For example, the product of $a^2+2ax-3x^2$ into $2a-x$, is obtained as follows.

$$\begin{array}{l} 1+2-3 \\ 2-1 \\ \hline 2+4-6 \\ \quad -1-2+3 \\ \hline 2+3-8+3 \end{array}$$

The product sought, $2a^3+3a^2x-8ax^2+3x^3$

It will sometimes happen that the multiplier and multiplicand are of the same form as in the preceding examples, except that one or more of the terms is wanting. In such cases, 0 may be put in the place of the co-efficient of each absent term, and the multiplication be then performed as above.

Suppose the factors are a^2-2b^2, $a^3-ab^2+3b^3$; the first of which is incomplete for want of a term containing ab; and the last, for want of a term containing a^2b.

If we were to multiply here as in the previous cases, the co-efficients would not all fall in their proper places, and we should be led into error. But this difficulty will be obviated, if we imagine $0\times ab$ and $0\times a^2b$ to stand in the place of the absent terms, and employ the co-efficient 0, in the same way with other co-efficients. The factors, without being changed in value, will thus become,

$$a^2+0ab-2b^2,\ a^3+0a^2b-ab^2+3b^3,$$

and the multiplication will proceed as follows,

Co-efficients of one factor,	$1+0-1+3$
Co-efficients of the other,	$1+0-2$
	$1+0-1+3$
	$\quad -2-0+2-6$
Co-efficients of the product,	$1+0-3+3+2-6$
Product,	$a^5-3a^3b^2+3a^2b^3+2ab^4-6b^5$

There is no term here containing a^4b, because the corresponding co-efficient is 0.

Examples.

1. Multiply $b^3-b^2x+bx^2-x^3$ by $b+x$.
2. Multiply y^3-3y^2+3y-1 by y^2-1.
3. Multiply $x^4+2x^2y^2+y^4$ by $x^2-2xy+y^2$.
4. Multiply a^2-1 by a^2+1.

109. The following theorems relate to certain cases in Multiplication of frequent occurrence, and should be carefully learned.

THEOREM I.

The square of the Sum of two quantities is equal to the square of the first, plus twice the product of the first and second, plus the square of the second.

This may be expressed algebraically thus,

$$(a+b)^2=a^2+2ab+b^2,$$

where a and b represent the two quantities, and $a+b$ is their sum. In proof of the theorem, it is sufficient to observe that $(a+b)^2$ is the same as $(a+b)\ (a+b)$; which, when expanded, becomes $a^2+2ab+b^2$.

From this, the square of the sum of any two quantities may be at once obtained, without multiplication.

Examples.

1. $(2a+xy)^2=4a^2+4axy+x^2y^2$.
2. $(b^2+3y)^2=b^4+6b^2y+9y^2$.
3. $(3x^3+1)^2=$
4. $(100+1)^2=$
5. $(90+5)^2=$
6. $(5a^2y+2ax^2)^2=$
7. $(4an^3+7)^2=$
8. $(12+\frac{1}{2})^2=$

Learners, not familiar with this theorem, are apt to assume the square of $a+b$ to be simply a^2+b^2.

THEOREM II.

110. *The square of the Difference of two quantities is equal to the square of the first, minus twice the product of the first and second, plus the square of the second.*

For, if the quantities are represented by a and b, their difference is $a-b$; and the square of this will, by multiplying, be found to be $a^2-2ab+b^2$: therefore, (to express the theorem algebraically,)

$$(a-b)^2=a^2-2ab+b^2.$$

Examples.

1. $(a-2x)^2=a^2-4ax+4x^2$.
2. $(100-1)^2=10000-200+1$, or $99^2=9801$.
3. $(1000-3)^2=$
4. $(5x^2-2y)^2=$
5. $(1-2x^3)^2=$
6. $(3a^2b-4ax^3)^2=$
7. $(20-\frac{1}{10})^2=$
8. $(2x^2-\frac{1}{2})^2=$
9. $(\frac{1}{10}-\frac{1}{100})^2=$

Learners often assume the square of $a-b$ to be a^2-b^2, instead of $a^2-2ab+b^2$, which is here seen to be the true expression.

THEOREM III.

111. *The product of the sum and difference of two quantities, is equal to the difference of their squares.*

For, if the quantities are a and b, $a+b$ is their sum, $a-b$ their difference; and the product of $a+b$ into $a-b$, as will be found by multiplying, is a^2-b^2.

The theorem may be stated algebraically thus,

$$(a+b)\,(a-b)=a^2-b^2.$$

Examples.

1. $(2x+1)\,(2x-1)=4x^2-1$.
2. $(3a+2x^2)\,(3a-2x^2)=9a^2-4x^4$.
3. $(a^2x+7x^2y)\,(a^2x-7x^2y)=$
4. $(2x^3y+1)\,(2x^3y-1)=$
5. $(1+4a^3)(1-4a^3)=$
6. $(x^2+7xy)\,(x^2-7xy)=$
7. $(10+\frac{1}{10})\,(10-\frac{1}{10})=$
8. $(100+7)\,(100-7)=$

SECTION V.

DIVISION.

Art. **112.** In multiplication, we have two factors given, and are required to find their product. By multiplying the factors 4 and 6, we obtain the product 24. But it is frequently necessary to reverse this process. The number 24, and *one* of the factors may be given, to enable us to find the other. The operation by which this is effected, is called *Division*. We obtain the number 4, by dividing 24 by 6. The quantity to be divided is called the dividend; the *given* factor, the divisor; and that which is *required*, the quotient.

113. Division *is finding a quotient, which multiplied into the divisor will produce the dividend.**

In multiplication the *multiplier* is always a *number*. (Art. 86.) And the *product* is a quantity of the same kind, as the multiplicand. (Art. 87.) The product of 3 rods into 4, is 12 rods. When we come to division, the product and *either* of the factors may be given, to find the other; that is,

The divisor may be a *number*, and then the quotient will be a quantity of the same kind as the dividend; or,

The *divisor* may be a quantity of the same kind as the dividend; and then the *quotient* will be a number.

Thus 12 *rods*÷3=4 *rods*. But 12 *rods*÷3 *rods*=4.

And 12 *rods*÷24=$\frac{1}{2}$ *rod*. And 12 *rods*÷24 *rods*=$\frac{1}{2}$.

In the first case, the divisor being a *number*, shows into *how many* parts the dividend is to be separated; and the quotient shows what these parts are.

If 12 rods be divided into 3 parts, each will be 4 rods long. And if 12 rods be divided into 24 parts, each will be *half* a rod long.

In the other case, if the divisor is *less* than the dividend, the former shows into *what* parts the latter is to be divided;

* The *remainder* is here supposed to be included in the quotient, as is commonly the case in algebra.

and the quotient shows *how many* of these parts are contained in the dividend. In other words, division in this case consists in finding *how often one quantity is contained in another.*

A line of 3 rods, is contained in one of 12 rods, *four times.*

But if the divisor is *greater* than the dividend, and yet a quantity of the same kind, the quotient shows *what part* of the divisor is equal to the dividend.

Thus *one half* of 24 rods is equal to 12 rods.

Division of Monomials.

114. As the product of the divisor and quotient is equal to the dividend, the quotient may be found, by resolving the dividend into two such factors, that one of them shall be the divisor. The other will, of course, be the quotient.

Suppose abd is to be divided by a. The factors a and bd will produce the dividend. The first of these, being a divisor, may be set aside. The other is the quotient. Hence,

When the divisor is found as a factor in the dividend, the division is performed by cancelling this factor.

Divide	cx	dh	d^2rx	hm^2y	$dhxy^2$	ab^4cd	$abxy$
By	c	d	d^2r	hm^2	dy^2	b^4	ax
Quot.	x		x		hx		by

In each of these examples, the letters which are common to the divisor and dividend, are set aside, and the other letters form the quotient. It will be seen at once, that the product of the quotient and divisor is equal to the dividend.

115. *A* power *is divided by another power of the same letter, by subtracting the exponent of the divisor from that of the dividend.*

Thus the quotient of a^5 divided by a^2 is a^3. For a^3 multiplied into a^2 will produce a^5. See Art. 92 and 113.

Also, $b^7 \div b = b^6$, $x^{10} \div x^8 = x^2$, $y^6 \div y^5 = y$.

116. In performing multiplication, if the factors contain *numeral figures*, these are multiplied into each other. (Art. 93.) Thus $3a$ into $7b$ is $21ab$. Now if this process is to be *re-*

versed, it is evident that dividing the number in the product, by the number in one of the factors, will give the number in the other factor. The quotient of $21ab \div 3a$ is $7b$. Hence,

In division, if there are numeral CO-EFFICIENTS *prefixed to the letters, the co-efficient of the dividend must be divided, by the co-efficient of the divisor.*

Divide	$6a^3b^7$	$24h^2x^3y^5$	$25dhr$	$20hn^2$	$34drx$	$12my^2$
By	$2a^2b^3$	$6hxy^4$	dh	5	34	y
Quot.	$3a\ b^4$		$25r$		drx	

117. *In division, the same rule is to be observed respecting the* SIGNS, *as in multiplication; that is, if the divisor and dividend are both positive, or both negative, the quotient must be positive; if one is positive and the other negative, the quotient must be negative.* (Art. 99.)

This is manifest from the consideration that the product of the divisor and quotient must be the same as the dividend.

$$\text{If}\ \left.\begin{array}{r} +a\times+b=+ab \\ -a\times+b=-ab \\ +a\times-b=-ab \\ -a\times-b=+ab \end{array}\right\}\ \text{then}\ \left\{\begin{array}{l} +ab\div+b=+a \\ -ab\div+b=-a \\ -ab\div-b=+a \\ +ab\div-b=-a \end{array}\right.$$

Div.	abx	$8a\times10ay$	$-3a^3x\times6a^2$	$6am\times dh$
By	$-a$	$-2a$	$-3a^2$	$-2a$
Quot.	$-bx$	$-40ay$		$-3m\times dh=-3dhm$

5. Divide $2a^3bx^5y^4$ by $-a^3bxy^2$.

6. Divide $-33a^2bn^3x^5$ by $-3an^2x^3$.

118. *If the letters of the divisor are not to be found in the dividend, the division is expressed by writing the divisor under the dividend, in the form of a vulgar fraction.*

Thus $xy\div a=\dfrac{xy}{a}$; and $5a\div-h=\dfrac{5a}{-h}$.

This is a method of *denoting* division, rather than an actual performing of the operation. But the purposes of division may frequently be answered, by these fractional expressions. As they are of the same nature with other vulgar fractions, they may be added, subtracted, multiplied, &c. See the next section.

Division of Polynomials.

119. When a simple factor is multiplied into a *compound* one, the former enters into *every* term of the latter. (Art. 95.) Thus a into $b+d$, is $ab+ad$. Such a product is easily resolved again into its original factors.

Thus $ab+ad=a\times(b+d)$.

$ab+ac+ah=a\times(b+c+h)$.

$amh+amx+amy=am\times(h+x+y)$.

$4ad+8ah-12am+4ay=4a\times(d+2h-3m+y)$.

Now if the whole quantity be divided by one of these factors, the quotient will be the other factor. See Art. 114.

Thus, $(ab+ad)\div a=b+d$. Hence,

If the divisor is contained in every term of a COMPOUND DIVIDEND, *it must be cancelled in each.*

Div.	$ab+ac$	$ahn-ahx$	a^2h-ay	$h^2y^2-ahy^3+hn^2y^2$
By	a	ah	a	hy^2
Quot.	$b+c$		$ah-y$	

And if there are *co-efficients*, these must be divided, in each term also.

Div.	$6ab-12a^2c$	$12d^2x-15a^2d^3y$	$12hx+8$	$15xy^2-5x^2$
By	$3a$	$3d^2$	-4	$5x$
Quot.	$2b\ -4ac$		$-3hx-2$	

120. When the divisor is not contained in all the terms of the dividend, the division may be expressed, either by placing the divisor under the *whole* dividend, or by repeating it under *each term*, taken separately. There are occasions

when it will be convenient to exchange one of these forms of expression for the other.

Thus $b+c$ divided by x, is either $\frac{b+c}{x}$, or $\frac{b}{x}+\frac{c}{x}$.

And $a+b$ divided by 2, is either $\frac{a+b}{2}$, that is, half the sum of a and b; or $\frac{a}{2}+\frac{b}{2}$, that is, the sum of half a and half b.

For it is evident that *half the sum* of two or more quantities, is equal to the *sum of their halves.* And the same principle is applicable to a third, fourth, fifth, or any other portion of the dividend.

So also $a-b$ divided by 2, is either $\frac{a-b}{2}$, or $\frac{a}{2}-\frac{b}{2}$.

For *half the difference* of two quantities is equal to the *difference of their halves.*

So $\frac{a-2b+h}{m}=\frac{a}{m}-\frac{2b}{m}+\frac{h}{m}$. And $\frac{3a-c}{-x}=\frac{3a}{-x}-\frac{c}{-x}$.

121. If *some* of the letters in the divisor are in each term of the dividend, the fractional expression may be rendered more simple, by rejecting equal factors from the numerator and denominator.

Div.	ab	n^2xy	$ahm-3ay$	$hx+h^2n$	$2am$
By	ac	hn	ab	hy	$2xy$
Quot.	$\frac{ab}{ac}$ or $\frac{b}{c}$		$\frac{hm-3y}{b}$		$\frac{am}{xy}$

These reductions are made upon the principle, that a given divisor is contained in a given dividend, just as many times, as double the divisor in double the dividend; triple the divisor in triple the dividend; &c. See the reduction of fractions.

122. If the divisor is in some of the terms of the dividend, but not in all; those which contain the divisor may be divided as in Art. 114, and the others set down in the form of a fraction.

Thus $(ab+d)\div a$ is either $\frac{ab+d}{a}$, or $\frac{ab}{a}+\frac{d}{a}$, or $b+\frac{d}{a}$.

Div.	$dxy+rx-hd$	$3x^2y+nx+b$	$bm+3y$	$5a^2b-ax$
By	x	x	$-b$	$5a$
Quot.	$dy\ +r-\frac{hd}{x}$		$-m+\frac{3y}{-b}$	

123. The quotient of any quantity divided by *itself* or *its equal*, is obviously *a unit*.

Thus $\frac{a}{a}=1$. And $\frac{3ax}{3ax}=1$. And $\frac{6}{4+2}=1$. And $\frac{a+b-3h}{a+b-3h}=1$.

Div.	$ax+x$	$6ax^2+2x$	$4axy-4a+8a^2d$	$8a^2y-4x^2-4$
By	x	$-2x$	$4a$	4
Quot.	$a+1$		$xy-1\ +2ad$	

Cor. If the dividend is *greater* than the divisor, the quotient must be *greater than a unit:* But if the dividend is *less* than the divisor, the quotient must be *less than a unit.*

124. A general rule for division by *compound divisors* is given in Art. 126. One case, however, deserves particular notice.

From what is stated at the beginning of Art. 119, it is obvious, that

If a compound expression containing any factor in every term, be divided by the other quantities connected by their signs, the quotient will be that factor.

Div.	$ab+ac+ah$	$ax^2y+hx^2y+mx^2y$	$6a^2y-2axy$	$2ab-4ax$
By	$b+\ c+\ h$	$a+h+m$	$3a-x$	$b-2x$
Quot.	a		$2ay$	

125. In division as well as in multiplication, the caution must be observed, not to confound *terms* with *factors*. See Art. 96.

Thus $ab\times ac\div a=a^2bc\div a=abc$.

But $(ab+ac)\div a=b+c$. (Art. 119.)

And $ab\times ac\div b\times c=a^2bc\div bc=a^2$.

But $(ab+ac)\div(b+c)=a$. (Art. 124.)

It is a common mistake of beginners, to suppose that the quotient of $ab+ac$ divided by $b+c$, is $a+a$ instead of a.

126. *When the* DIVISOR *and* DIVIDEND *are both* POLYNOMIALS, the *general rule* for performing the division is the following; which is substantially the same, as the rule for division in arithmetic:

*To obtain the first term of the quotient, divide the first term of the dividend, by the first term of the divisor;**

Multiply the whole divisor, by the term placed in the quotient; subtract the product from a part of the dividend; and to the remainder bring down as many of the following terms, as shall be necessary to continue the operation:

Divide again by the first term of the divisor, and proceed as before, till all the terms of the dividend are brought down.

Ex. 1. Divide $ac+bc+ad+bd$, by $a+b$.

	$ac+bc+ad+bd$	$a+b$	the divisor.
the first subtrahend	$ac+bc$	$c+d$	the quotient.
	$*\quad *\quad ad+bd$		
the second subtrahend	$ad+bd$		
	$*\quad *$		

Here ac, the first term of the dividend, is divided by a, the first term of the divisor, (Art. 114.) which gives c for the first term of the quotient. Multiplying the whole divisor by this, we have $ac+bc$ to be subtracted from the two first terms of the dividend. The two remaining terms are then brought down, and the first of them is divided by the first term of the divisor as before. This gives d for the second term of the quotient. Then multiplying the divisor by d, we have $ad+bd$ to be subtracted, which exhausts the whole dividend, without leaving any remainder.

The rule is founded on this principle, that the product of the divisor into the several parts of the quotient, is equal to the dividend. (Art. 113.) Now by the operation, the product of the divisor into the *first* term of the quotient is sub-

---

* See Note C.

tracted from the dividend; then the product of the divisor into the *second* term of the quotient; and so on, till the product of the divisor into each term of the quotient, that is, the product of the divisor into the *whole* quotient, (Art. 97.) is taken from the dividend. If there is no remainder, it is evident that this product is *equal* to the dividend. If there *is* a remainder, the product of the divisor and quotient is equal to the whole of the dividend *except* the remainder. And this remainder is not included in the parts subtracted from the dividend, by operating according to the rule.

The divisor is sometimes set at the left of the dividend, but it is more convenient to place it on the right.

127. Before beginning to divide, it will generally be expedient to make some preparation in the *arrangement of the terms.*

The letter which is in the first term of the divisor, should be in the first term of the dividend also. And the *powers* of this letter should be arranged in order, both in the divisor and in the dividend; the highest power standing first, the next highest next, and so on.

Ex. 2. Divide $2a^2b+b^3+2ab^2+a^3$, by a^2+b^2+ab.

Here, if we take a^2 for the first term of the divisor, the other terms should be arranged according to the powers of a, thus,

$$\begin{array}{l|l}
a^3+2a^2b+2ab^2+b^3 & a^2+ab+b^2 \\
\cline{2-2}
a^3+\ a^2b+\ ab^2 & a+b \\
\cline{1-1}
\qquad a^2b+\ ab^2+b^3 & \\
\qquad a^2b+\ ab^2+b^3 & \\
\cline{1-1}
\qquad * \quad * \quad * &
\end{array}$$

In these operations, particular care will be necessary in the management of *negative quantities.* Constant attention must be paid to the rules for the signs in subtraction, multiplication and division. (Arts. 75, 99, 117.)

Ex. 3. Divide $2ax-2a^2x-3a^2xy+6a^3x+axy-xy$, by $2a-y$.

If the terms be arranged according to the powers of a, they will stand thus;

$$
\begin{array}{l|l}
6a^3x-3a^2xy-2a^2x+axy+2ax-xy & 2a-y \\
\cline{2-2}
6a^3x-3a^2xy & 3a^2x-ax+x \\
\cline{1-1}
\quad * \qquad * \quad -2a^2x+axy & \\
\qquad\qquad\quad\; -2a^2x+axy & \\
\cline{1-1}
\qquad\qquad\quad\; * \qquad * \quad +2ax-xy & \\
\qquad\qquad\qquad\qquad\qquad\;\; +2ax-xy & \\
\cline{1-1}
\end{array}
$$

Ex. 4. Divide $7xy^2+ax^3+x^4-7y^2-ax^2-x^3$, by $x-1$.
Quot. $x^3+ax^2+7y^2$.

Ex. 5. Divide $x+6a^2h+2a^2x-4a^2n+3h-2n$,
by $3h-2n+x$. Quot. $2a^2+1$.

Ex. 6. Divide $7xy^3+5x^3y-6y^4+3x^4-9x^2y^2$,
by $2xy+x^2-3y^2$. Quot. $3x^2-xy+2y^2$.

128. In multiplication, some of the terms, by balancing each other, may be lost in the product. (Art. 104.) These may *re-appear* in division, so as to present terms, in the course of the process, different from any which are in the dividend.

Ex. 7.

$$
\begin{array}{l|l}
a^3+x^3 & a+x \\
\cline{2-2}
a^3+a^2x & a^2-ax+x^2 \\
\cline{1-1}
\quad -a^2x+x^3 & \\
\quad -a^2x-ax^2 & \\
\cline{1-1}
\qquad * \quad ax^2+x^3 & \\
\qquad\quad\;\; ax^2+x^3 & \\
\cline{1-1}
\end{array}
$$

Ex. 8.

$$
\begin{array}{l|l}
a^4+4x^4 & a^2-2ax+2x^2 \\ \cline{2-2}
a^4-2a^3x+2a^2x^2 & a^2+2ax+2x^2 \\ \cline{1-1}
* \ +2a^3x-2a^2x^2+4x^4 & \\
\quad +2a^3x-4a^2x^2+4ax^3 & \\ \cline{1-1}
\quad * \quad +2a^2x^2-4ax^3+4x^4 & \\
\quad\quad\quad +2a^2x^2-4ax^3+4x^4 & \\ \cline{1-1}
\end{array}
$$

If the learner will take the trouble to multiply the quotient into the divisor, in the two last examples, he will find in the partial products, the several terms which appear in the process of dividing. But most of them, by balancing each other, are lost in the general product.

129. When there is a *remainder* after all the terms of the dividend have been brought down, this may be placed over the divisor and added to the quotient, as in arithmetic.

Ex. 9.

$$
\begin{array}{l|l}
ac+bc+ad+bd+x & a+b \\ \cline{2-2}
ac+bc & c+d+\dfrac{x}{a+b} \\ \cline{1-1}
* \quad * \quad ad+bd & \\
\qquad\quad ad+bd & \\ \cline{1-1}
\qquad\quad * \quad * \quad x & \\ \cline{1-1}
\end{array}
$$

Ex. 10.

$$
\begin{array}{l|l}
ad-ah+bd-bh+y & d-h \\ \cline{2-2}
ad-ah & a+b+\dfrac{y}{d-h} \\ \cline{1-1}
* \quad * \quad bd-bh & \\
\qquad\quad bd-bh & \\ \cline{1-1}
\qquad\quad * \quad * \quad y & \\ \cline{1-1}
\end{array}
$$

It is evident that $a+b$ is the quotient belonging to the whole of the dividend, *excepting* the remainder y. (Art. 126.)

And $\frac{y}{d-h}$ is the quotient belonging to this remainder. (Art. 118.)

Ex. 11. Divide $3ab-6ax-bh+2hx+2b^2-4bx+c$, by $b-2x$. Quot. $3a-h+2b+\frac{c}{b-2x}$.

Ex. 12. Divide $6a^3+4a^2-3ax+3a-2x+7$, by $3a+2$. Quot. $2a^2-x+1+\frac{5}{3a+2}$.

Promiscuous Examples.

1. Divide $5x^2y-10ax^3+15bxy^2-5x$, by $5x$.
2. Divide $12h-15a+3y^2-3-9bcy+2x$, by 3.
3. Divide $(2+ax)(3-c)y$, by $(2+ax)(3-c)$.
4. Divide $x^2y-3xy-x+2xy^2$, by $xy-3y-1+2y^2$.
5. Divide $b^2h+nx-by+2a^2h-3-b$, by $-b$.
6. Divide $bnx-2n^2-ahn-4n+x$, by $-n^2x$.
7. Divide $2b^2-bx^2y-3bx+x-9$, by $3bx^2y$.
8. Divide $4a^3x^2-a^2+6-2a^2x-2$, by $-2a^3$.
9. Divide $b+x+bnx+nx^2$, by $b+x$. Quot. $1+nx$.
10. Divide $1-3x+3x^2-x^3$, by $1-x$.
11. Divide $b^3+3b^2x+3bx^2+x^3$, by $b+x$.
12. Divide $3x^3+16x^2-5x+11$, by $x+5$.
13. Divide $1-a^5$, by $1-a$.
14. Divide $x^4-6x^2y^2+5y^4$, by $x-3y$.
15. Divide $6a^4-10a^3+4a-15$, by $3a^2-2a+1$.

130. A regular series of quotients is obtained, by dividing the difference of the *powers* of two quantities, by the difference of the quantities. Thus,

$$(y^2-a^2)\div(y-a)=y+a,$$
$$(y^3-a^3)\div(y-a)=y^2+ay+a^2,$$
$$(y^4-a^4)\div(y-a)=y^3+ay^2+a^2y+a^3,$$
$$(y^5-a^5)\div(y-a)=y^4+ay^3+a^2y^2+a^3y+a^4,$$
&c.

Here it will be seen, that the index of y, in the first term of the quotient, is less by 1, than in the dividend; and that it decreases by 1, from the first term to the last but one: While the index of a, increases by 1, from the second term to the last, where it is less by 1, than in the dividend.

This may be expressed in a general formula, thus,

$$(y^m-a^m)\div(y-a)=y^{m-1}+ay^{m-2}\ \ldots\ +a^{m-2}y+a^{m-1}.$$

To demonstrate this, we have only to multiply the quotient into the divisor. (Art. 113.)

All the terms except two, in the partial products, will be balanced by each other; and will leave the general product the same as the dividend.

$$\begin{array}{ll}
\text{Mult.} & y^4+ay^3+a^2y^2+a^3y+a^4 \\
\text{Into} & y\ -a \\ \hline
 & y^5+ay^4+a^2y^3+a^3y^2+a^4y \\
 & \quad\ -ay^4-a^2y^3-a^3y^2-a^4y-a^5 \\ \hline
\text{Prod.} & y^5\quad *\quad *\quad *\quad *\ -a^5.
\end{array}$$

$$\begin{array}{ll}
\text{Mult.} & y^{m-1}+ay^{m-2}+a^2y^{m-3}\ldots+a^{m-2}y+a^{m-1} \\
\text{Into} & y-a \\ \hline
 & y^m+ay^{m-1}+a^2y^{m-2}\ldots+a^{m-2}y^2+a^{m-1}y \\
 & \quad\ -ay^{m-1}-a^2y^{m-2}\ldots-a^{m-2}y^2-a^{m-1}y-a^m \\ \hline
\text{Prod.} & y^m\quad *\quad *\quad *\quad *\ -a^m.
\end{array}$$

131. In the same manner it may be proved, that the difference of the powers of two quantities, if the index is an *even* number, is divisible by the *sum* of the quantities. That is, as the double of every number is even;

$$(y^{2m}-a^{2m})\div(y+a)=y^{2m-1}-ay^{2m-2}\ldots+a^{2m-2}y-a^{2m-1}.$$

And the *sum of the powers* of two quantities, if the index is an *odd* number, is divisible by the *sum of the quantities*. That is, as $2m+1$ is an odd number;

$$(y^{2m+1}+a^{2m+1})\div(y+a)=y^{2m}-ay^{2m-1}\ldots-a^{2m-1}y+a^{2m}.$$

For in each of these cases the product of the quotient and divisor, is equal to the dividend.

Thus,

$$(y^2-a^2)\div(y+a)=y-a,$$
$$(y^4-a^4)\div(y+a)=y^3-ay^2+a^2y-a^3,$$
$$(y^6-a^6)\div(y+a)=y^5-ay^4+a^2y^3-a^3y^2+a^4y-a^5,$$

&c.

And

$$(y^3+a^3)\div(y+a)=y^2-ay+a^2,$$
$$(y^5+a^5)\div(y+a)=y^4-ay^3+a^2y^2-a^3y+a^4,$$
$$(y^7+a^7)\div(y+a)=y^6-ay^5+a^2y^4-a^3y^3+a^4y^2-a^5y+a^6,$$

&c.

Division by Detached Co-efficients.

132. Division as well as Multiplication, may sometimes be conveniently performed by means of detached co-efficients.

Suppose that $2a^4+11a^3x+20a^2x^2+13ax^3+2x^4$ is to be divided by $a^2+3ax+2x^2$.

To perform the division in the usual way, we proceed as follows,

$$\begin{array}{l|l}
2a^4+11a^3x+20a^2x^2+13ax^3+2x^4 & a^2+3ax+2x^2 \\
\cline{2-2}
2a^4+\ \ 6a^3x+\ \ 4a^2x^2 & 2a^2+5ax+\ x^2\text{, Quot.} \\
\cline{1-1}
\qquad 5a^3x+16a^2x^2+13ax^3 & \\
\qquad 5a^3x+15a^2x^2+10ax^3 & \\
\cline{1-1}
\qquad\qquad\quad a^2x^2+\ \ 3ax^3+2x^4 & \\
\qquad\qquad\quad a^2x^2+\ \ 3ax^3+2x^4 & \\
\cline{1-1}
\end{array}$$

But here, as the order in which the letters and exponents occur is obvious, we may omit them, and proceed with the detached co-efficients, thus;

$$\begin{array}{ll|ll}
\text{Co-eff. of div'd.} & 2+11+20+13+2 & 1+3+2 & \text{co-eff. of div'r.} \\
\cline{3-3}
 & 2+\ 6+\ 4 & 2+5+1 & \text{co-eff. of quot.} \\
\cline{2-2}
 & \quad 5+16+13 & & \\
 & \quad 5+15+10 & 2a^2+5ax+x^2\text{, Quot.} & \\
\cline{2-2}
 & \qquad 1+\ 3+2 & & \\
 & \qquad 1+\ 3+2 & & \\
\cline{2-2}
\end{array}$$

After obtaining the co-efficients of the quotient by dividing as above, the proper letters and exponents are readily supplied. As the first term of the divisor is a^2, and the first term of the dividend $2a^4$, the first term of the quotient must be $2a^2$; and hence it is evident that the letters and exponents in the following terms must be ax and x^2.

For the co-efficients of terms that are wanting in the divisor or dividend, ciphers must be substituted. For example,

$a^5-3a^3b^2+3a^2b^3+2ab^4-6b^5$, in which there is no term containing a^4b, is divided by $a^3-ab^2+3b^3$, in which there is no term containing a^2b, as follows;

$$\begin{array}{l|l}
1+0-3+3+2-6 & 1+0-1+3 \\
\underline{1+0-1+3\qquad\quad} & \overline{1+0-2}\quad \text{co-eff. of quot.} \\
\quad 0-2+0+2-6 & \\
\quad\ \underline{-2+0+2-6} & a^2-2b^2,\ \text{Quotient.}
\end{array}$$

It will be seen that the second co-efficient of the quotient must be 0, because the first remainder begins with 0, under the second term of the dividend. And in passing to the third co-efficient, we must annex -6, as well as $+2$, to the first remainder.

Examples.

1. Divide $a^4-a^2x^2+2ax^3-x^4$, by a^2-ax+x^2.
2. Divide $x^4-4x^3+6x^2-4x+1$, by $x-1$.
3. Divide $x^6+x^4y-5x^3y^2+6xy^4+2y^6$, by $x^2+3xy+y^2$.
4. Divide $1-4b+10b^2-16b^3+17b^4-12b^5$, by $1-2b+3b^2$.

133. From the nature of division it is evident, that the value of the quotient depends both on the divisor and the dividend. With a given divisor, the greater the dividend, the greater the quotient. And with a given dividend, the greater the divisor, the less the quotient. In several of the succeeding parts of algebra, particularly the subjects of fractions, ratios, and proportion, it will be important to be able to determine what change will be produced in the quotient, by increasing or diminishing either the divisor or the dividend.

If the given dividend be 24, and the divisor 6; the quotient will be 4. But this same dividend may be supposed to be multiplied or divided by some *other* number, before it is divided by 6. Or the *divisor* may be multiplied or divided by some other number, before it is used in dividing 24. In each of these cases, the quotient will be altered.

134. In the first place, if the given divisor is contained in the given dividend a certain number of times, it is obvious that the same divisor is contained,

In *double* that dividend, *twice* as many times;

In *triple* the dividend, *thrice* as many times; &c.

That is, if the divisor remains the same, *multiplying the dividend* by any quantity, is, in effect, *multiplying the quotient* by that quantity.

Thus, if the constant divisor is 6, then $24 \div 6 = 4$ the quotient.

Multiplying the dividend by 2, $2 \times 24 \div 6 = 2 \times 4$

Multiplying by any number n, $n \times 24 \div 6 = n \times 4$

135. Secondly, if the given divisor is contained in the given dividend a certain number of times, the same divisor is contained,

In *half* that dividend, half as many times;

In *one third* of the dividend, one third as many times; &c.

That is, if the divisor remains the same, *dividing the dividend* by any quantity, is, in effect, *dividing the quotient* by that quantity.

Thus, $24 \div 6 = 4$

Dividing the dividend by 2, $\frac{1}{2}24 \div 6 = \frac{1}{2}4$

Dividing by n, $\frac{1}{n}24 \div 6 = \frac{1}{n}4$

136. Thirdly, if the given divisor is contained in the given dividend a certain number of times, then, in the same dividend,

Twice that divisor is contained only *half* as many times;

Three times the divisor is contained *one third* as many times.

That is, if the dividend remains the same, *multiplying the divisor* by any quantity, is, in effect, *dividing the quotient* by that quantity.

Thus $24 \div 6 = 4$

Multiplying the divisor by 2, $24 \div 2 \times 6 = \frac{4}{2}$

Multiplying by n, $24 \div n \times 6 = \frac{4}{n}$

137. Lastly, if the given divisor is contained in the given dividend a certain number of times, then, in the same dividend,

Half that divisor is contained *twice* as many times;

One third of the divisor is contained *thrice* as many times.

That is, if the dividend remains the same, *dividing the divisor* by any quantity is, in effect, *multiplying the quotient* by that quantity.

Thus	$24 \div 6 = 4$
Dividing the divisor by 2,	$24 \div \frac{1}{2}6 = 2 \times 4$
Dividing by n,	$24 \div \frac{1}{n}6 = n \times 4$

Resolving Polynomials into Factors.

138. Polynomials may often be rendered more convenient for certain purposes, by resolving them into factors.

It is, in many cases, easy to discover the factors by inspection.

Examples.

1. Resolve $h^3+3h^2x-3hy^2$ into factors.

Ans. $h(h^2+3hx-3y^2)$. See Art. 119.

2. Resolve $4a^2-9x^2$ into factors.

Ans. $(2a+3x)(2a-3x)$. See Art. 111.

3. Resolve $x^2+6xy+9y^2$ into factors.

Ans. $(x+3y)(x+3y)$. See Art. 109.

4. Resolve $25a^2-10ab+b^2$ into factors.

Ans. $(5a-b)(5a-b)$. See Art. 110.

5. Resolve $4x^3+12x^2y+9xy^2$ into three factors.

6. Resolve x^3+y^3 into factors.

7. Resolve $1-a^3$ into factors.

8. Resolve a^4-b^4 into its factors.

9. Resolve y^6+1 into factors.

10. Resolve $4x^3-4x^2+x$ into its factors.

11. Resolve $abh^2-2abh+ab$ into its factors.

12. Resolve $63nx^2y^2+84nxy+28n$ into its factors.

13. Resolve $9h-h$ into three factors.

14. Resolve 3^8-1 into four factors.

15. Resolve $25h^2n^4-\frac{1}{4}$ into factors.

SECTION VI.

FRACTIONS.

ART. **139.** Expressions in the form of *fractions* occur more frequently in algebra than in arithmetic. Most instances in division belong to this class. Indeed the numerator of *every* fraction may be considered as a *dividend*, of which the denominator is a *divisor*.

According to the common definition in arithmetic, the denominator shows into what parts an integral unit is supposed to be divided; and the numerator shows how many of these parts belong to the fraction. But it makes no difference, whether the *whole* of the numerator is divided by the denominator; or only *one* of the integral units is divided, and then the quotient taken as many times as the number of units in the numerator. Thus $\frac{3}{4}$ is the same as $\frac{1}{4}+\frac{1}{4}+\frac{1}{4}$. A fourth part of *three* dollars, is equal to three fourths of *one* dollar.

140. The *value* of a fraction, is the *quotient* of the numerator divided by the denominator.

Thus the value of $\dfrac{6}{2}$ is 3. The value of $\dfrac{ab}{b}$ is a.

From this it is evident, that whatever changes are made in the *terms* of a fraction; if the *quotient* is not altered, the value remains the same. For any fraction, therefore, we may substitute any *other* fraction which will give the same quotient.

Thus $\dfrac{4}{2}=\dfrac{10}{5}=\dfrac{4ba}{2ba}=\dfrac{8drx}{4drx}=\dfrac{6+2}{3+1}$, &c. For the quotient in each of these instances is 2.

By the *terms* of a fraction are meant the numerator and denominator. This use of the word *terms* is not to be confounded with the more common one, explained in Art. 38. Each term of a fraction may be a polynomial. Thus, one term of $\dfrac{a+b}{c+d}$ is $a+b$, and the other is $c+d$.

141. As the value of a fraction is the quotient of the numerator divided by the denominator, it is evident from Art. 123, that when the numerator is *equal* to the denominator, the value of the fraction is *a unit;* when the numerator is *less* than the denominator, the value is *less than a unit;* and when the numerator is *greater* than the denominator, the value is *greater than a unit.*

The calculations in fractions depend on a few general principles, which will here be stated in connexion with each other.

142. If the *denominator* of a fraction remains the *same,*

Multiplying the numerator by any quantity, is multiplying the value by that quantity; and dividing the numerator, is dividing the value.

For the numerator and denominator are a dividend and divisor, of which the value of the fraction is the quotient. And by Art. 134 and 135, multiplying the dividend is in effect multiplying the quotient, and dividing the dividend is dividing the quotient.

Thus in the fractions $\frac{ab}{a}$, $\frac{3ab}{a}$, $\frac{7abd}{a}$, $\frac{\frac{1}{3}ab}{a}$, &c.

The quotients or values are b, $3b$, $7bd$, $\frac{1}{3}b$, &c.

Here it will be seen that, while the denominator is not altered, the value of the fraction is multiplied or divided by the same quantity as the numerator.

Cor. With a *given denominator,* the greater the numerator, the greater will be the value of the fraction; and, on the other hand, the greater the value, the greater the numerator.

143. If the *numerator* remains the *same,*

Multiplying the denominator by any quantity, is dividing the value by that quantity; and dividing the denominator, is multiplying the value.

For multiplying the divisor is dividing the quotient; and dividing the divisor is multiplying the quotient. (Art. 136, 137.)

In the fractions $\frac{24ab}{6b}$, $\frac{24ab}{12b}$, $\frac{24ab}{3b}$, $\frac{24ab}{b}$, &c.

The values are $4a$, $2a$, $8a$, $24a$, &c.

Cor. With a *given numerator*, the greater the denominator, the less will be the value of the fraction; and the less the value, the greater the denominator.

144. From the last two articles it follows, that *dividing the numerator* by any quantity, will have the same effect on the value of the fraction, as *multiplying the denominator* by that quantity; and *multiplying the numerator* will have the same effect, as *dividing the denominator.*

145. It is also evident from the preceding articles, that

If the numerator and denominator be both multiplied, or both divided, by the same quantity, the value of the fraction will not be altered.

Thus $\frac{bx}{b}=\frac{abx}{ab}=\frac{3bx}{3b}=\frac{\frac{1}{2}bx}{\frac{1}{2}b}=\frac{\frac{1}{3}abx}{\frac{1}{3}ab}$, &c. For in each of these instances the quotient is x.

146. An *integral quantity* may, without altering its value, be *converted into a fraction* having any proposed denominator, by multiplying the quantity into this denominator, and making the product the numerator.

Thus $a=\frac{a}{1}=\frac{ab}{b}=\frac{ad+ah}{d+h}=\frac{6adh}{6dh}$, &c. For the quotient of each of these is a.

So $d+h=\frac{dx+hx}{x}$. And $r+1=\frac{2dr^2+2dr}{2dr}$.

147. There is nothing, perhaps, in the calculation of algebraic fractions, which occasions more perplexity to a learner, than the positive and negative *signs*. The changes in these are so frequent, that it is necessary to become familiar with the principles on which they are made. The use of the sign which is prefixed to the dividing line, is to show whether the value of the *whole fraction* is to be added to, or subtracted from, the other quantities with which it is connected. (Art. 32.) This sign, therefore, has an influence on the several terms taken collectively. But in the numerator and denominator, each sign affects only the single term to which it is applied.

The value of $\frac{ab}{b}$ is a. (Art. 140.) But this will become negative, if the sign $-$ be prefixed to the fraction.

Thus $y+\frac{ab}{b}=y+a$. But $y-\frac{ab}{b}=y-a$.

So that changing the sign which is before the whole fraction, has the effect of changing the *value* from positive to negative, or from negative to positive.

Next, suppose the sign or signs of the *numerator* to be changed.

By Art. 117, $\frac{ab}{b}=+a$. But $\frac{-ab}{b}=-a$.

And $\frac{ab-bc}{b}=+a-c$. But $\frac{-ab+bc}{b}=-a+c$.

That is, by changing all the signs of the numerator, the value of the fraction is changed from positive to negative, or the contrary.

Again, suppose the sign of the *denominator* to be changed.

As before $\frac{ab}{b}=+a$. But $\frac{ab}{-b}=-a$.

We have then, this general proposition;

If the sign prefixed to a fraction, or all the signs of the numerator, or all the signs of the denominator be changed; the value of the fraction will be changed, from positive to negative, or from negative to positive.

148. From this is derived another important principle. As each of the changes mentioned here is from positive to negative, or the contrary; if any *two* of them be made at the same time, *they will balance each other.*

Thus by changing the sign of the numerator,

$$\frac{ab}{b}=+a \text{ becomes } \frac{-ab}{b}=-a.$$

But, by changing both the numerator and denominator, it becomes $\frac{-ab}{-b}=+a$, where the positive value is restored.

By changing the sign before the fraction,

$$y+\frac{ab}{b}=y+a \text{ becomes } y-\frac{ab}{b}=y-a.$$

But by changing the sign of the numerator also, it becomes $y-\frac{-ab}{b}$ where the quotient $-a$ is to be *subtracted* from y, or which is the same thing, (Art. 74.) $+a$ is to be *added*, making $y+a$ as at first. Hence,

If all the signs both of the numerator and denominator, or the signs of one of these with the sign prefixed to the whole fraction, be changed at the same time, the value of the fraction will not be altered.

Thus $\frac{6}{2}=\frac{-6}{-2}=-\frac{-6}{2}=-\frac{6}{-2}=+3.$

And $\frac{6}{-2}=\frac{-6}{2}=-\frac{6}{2}=-\frac{-6}{-2}=-3.$

Hence the quotient in division may be set down in different ways. Thus $(a-c)\div b$, is either $\frac{a}{b}+\frac{-c}{b}$, or $\frac{a}{b}-\frac{c}{b}$.

The latter method is the most common. See the examples in Art. 122.

Reduction of Fractions.

149. From the principles which have been stated, are derived the rules for the *reduction* of fractions, which are substantially the same in algebra, as in arithmetic.

A fraction may be REDUCED TO LOWER TERMS, *by dividing both the numerator and denominator, by any quantity which will divide them without a remainder.*

According to Art. 145, this will not alter the *value* of the fraction.

Thus $\frac{ab}{cb}=\frac{a}{c}$. And $\frac{6dm}{8dy}=\frac{3m}{4y}$. And $\frac{7m}{7mr}=\frac{1}{r}$.

In the last example, both parts of the fraction are divided by the numerator.

Again, $\frac{a+bc}{(a+bc)m}=\frac{1}{m}$. And $\frac{am+ay}{bm+by}=\frac{a}{b}$.

If a letter is in *every* term, both of the numerator and denominator, it may be *cancelled;* for this is *dividing* by that letter. (Art. 119.)

Thus, $\frac{3am+ay}{ad+ah}=\frac{3m+y}{d+h}$. $\frac{dry+dy}{dhy-dy}=\frac{r+1}{h-1}$.

If the numerator and denominator be divided by the *greatest common measure,* it is evident that the fraction will be reduced to the *lowest* terms. For the method of finding the greatest common measure, see Sect. XV.

150. *Fractions of different denominators may be* REDUCED TO A COMMON DENOMINATOR, *by multiplying each numerator into all the denominators except its own, for a new numerator; and all the denominators together, for a common denominator.*

Ex. 1. Reduce $\frac{a}{b}$, and $\frac{c}{d}$, and $\frac{m}{y}$, to a common denominator.

$$\left.\begin{array}{r} a\times d\times y=ady \\ c\times b\times y=cby \\ m\times b\times d=mbd \end{array}\right\}\text{ the three numerators.}$$

$$b\times d\times y=bdy \quad \text{the common denominator.}$$

The fractions reduced are $\frac{ady}{bdy}$, and $\frac{bcy}{bdy}$, and $\frac{bdm}{bdy}$.

Here it will be seen, that the reduction consists in multiplying the numerator and denominator of each fraction, into all the other denominators. This does not alter the value. (Art. 145.)

2. Reduce $\frac{ab}{2h}$, and $\frac{3n^2}{m}$, and $\frac{g}{3x}$.

3. Reduce $\frac{1}{4}$, and $\frac{2}{y^2}$, and $\frac{1-a}{x+2}$.

4. Reduce $\frac{2}{x+2}$, and $\frac{2}{x-2}$.

After the fractions have been reduced to a common denominator, they may be brought to lower terms, by the rule in the last article, if there is any quantity which will divide the denominator, and *all* the numerators without a remainder.

An *integer* and a fraction, are easily reduced to a common denominator. (Art. 146.)

Thus a and $\frac{b}{c}$ are equal to $\frac{a}{1}$ and $\frac{b}{c}$, or $\frac{ac}{c}$ and $\frac{b}{c}$.

And a, b, $\frac{h}{m}$, $\frac{d}{y}$ are equal to $\frac{amy}{my}$, $\frac{bmy}{my}$, $\frac{hy}{my}$, $\frac{dm}{my}$.

151. *To* REDUCE AN IMPROPER FRACTION *to an integral or a mixed quantity, divide the numerator by the denominator, for the integral part; and if there is a remainder, place this over the denominator for the fractional part.* See Art. 122.

Thus $\frac{ab+bm+d}{b}=a+m+\frac{d}{b}$.

Reduce $\frac{6xy-4x^2+2x-3h^2}{2x}$, to a mixed quantity.

For the reduction of a *mixed quantity* to an improper fraction, see Art. 154. And for the reduction of a *compound fraction* to a simple one, see Art. 163.

Addition of Fractions.

152. In adding fractions, we may either write them one after the other, with their signs, as in the addition of integers, or we may *incorporate them into a single fraction*, by the following *rule:*

Reduce the fractions to a common denominator, make the signs before them all positive, then add their numerators, and place the sum over the common denominator.

The common denominator shows into *what* parts the integral unit is supposed to be divided; and the numerators show the *number* of these parts belonging to each of the fractions. (Art. 139.) Therefore the numerators *taken together* show the whole number of parts in all the fractions

Thus, $\frac{2}{7}=\frac{1}{7}+\frac{1}{7}$. And $\frac{3}{7}=\frac{1}{7}+\frac{1}{7}+\frac{1}{7}$.

Therefore, $\frac{2}{7}+\frac{3}{7}=\frac{1}{7}+\frac{1}{7}+\frac{1}{7}+\frac{1}{7}+\frac{1}{7}=\frac{5}{7}$.

The numerators are added, according to the rules for the addition of integers. (Art. 60, &c.)

To avoid the perplexity which might be occasioned by the signs, it will be expedient to make those *prefixed* to the fractions uniformly positive, as the rule directs. But in doing this, care must be taken not to alter the value. This will be preserved, if all the signs in the numerator are changed at the same time with that before the fraction. (Art. 148.)

Ex. 1. Add $\frac{2}{16}$ and $\frac{4}{16}$ of a pound. Ans. $\frac{2+4}{16}$ or $\frac{6}{16}$.

It is as evident that $\frac{2}{16}$, and $\frac{4}{16}$ of a pound, are $\frac{6}{16}$ of a pound, as that 2 ounces and 4 ounces, are 6 ounces.

2. Add $\frac{a}{b}$ and $\frac{c}{d}$. First reduce them to a common denominator. They will then be $\frac{ad}{bd}$ and $\frac{bc}{bd}$, and their sum $\frac{ad+bc}{bd}$.

3. Given $\frac{m}{d}$ and $-\frac{2r+d}{3h}$, to find their sum.

Ans. $\frac{m}{d}$ and $-\frac{2r+d}{3h}=\frac{3hm}{3dh}$ and $-\frac{2dr+d^2}{3dh}=\frac{3hm-2dr-d^2}{3dh}$.

4. $\frac{b}{2x}$ and $-\frac{h^2-1}{a}=\frac{b}{2x}+\frac{-h^2+1}{a}=\frac{ab-2h^2x+2x}{2ax}$.

5. $\frac{2}{a}$ and $\frac{3b}{-x^2}=\frac{-2x^2}{-ax^2}+\frac{3ab}{-ax^2}=\frac{-2x^2+3ab}{-ax^2}$ or $\frac{2x^2-3ab}{ax^2}$.

6. $\frac{y}{y+x}$ and $\frac{x}{y-x}=\frac{y^2-xy+xy+x^2}{y^2-x^2}=\frac{y^2+x^2}{y^2-x^2}$. (Art. 70.)

7. Add $\frac{-h}{2y}$ to $\frac{-x^2}{1-n}$.

8. Add $\frac{-12}{3}$ to $\frac{-6}{7-5}$. Ans. -7.

153. For many purposes, it is sufficient to add fractions in the same manner as integers are added, by writing them one after another with their signs. (Art. 60.)

Thus the sum of $\frac{a}{b}$ and $\frac{3}{y}$ and $-\frac{d}{2m}$, is $\frac{a}{b}+\frac{3}{y}-\frac{d}{2m}$.

In the same manner, *fractions* and *integers* may be added.

The sum of a and $\frac{d}{y}$ and $3m$ and $-\frac{h}{r}$, is $a+3m+\frac{d}{y}-\frac{h}{r}$.

154. Or the integer may be *incorporated* with the fraction, by converting the former into a fraction, and then adding the numerators. See Art. 146.

The sum of a and $\frac{b}{m}$, is $\frac{a}{1}+\frac{b}{m}=\frac{am}{m}+\frac{b}{m}=\frac{am+b}{m}$.

The sum of $3y$ and $\frac{h+d}{m-2y}$, is $\frac{3my-6y^2+h+d}{m-2y}$.

Incorporating an integer with a fraction, is the same as *reducing a mixed quantity* to an improper fraction. For a mixed quantity is an integer and a fraction. In arithmetic, these are generally placed together, without any sign between them. But in algebra, they are distinct terms. Thus $2\frac{1}{3}$ is 2 *and* $\frac{1}{3}$, which is the same as $2+\frac{1}{3}$.

The *rule* for *reducing a mixed number* to an improper fraction, may be thus stated:

Multiply the integer into the denominator of the fraction, add the product to the numerator, and place the sum over the denominator.

If the sign before the fraction is $-$, it must, before applying the rule, be changed to $+$; and in order to leave the value of the fraction unaltered, the signs of the numerator or of the denominator must also be changed. See Art. 152.

Ex. 1. Reduce $2-\frac{1}{a}$. Ans. $\frac{2a-1}{a}$.

2. Reduce $a-1-\frac{3}{1-b}$. Ans. $\frac{a-ab+b-4}{1-b}$.

3. Reduce $2b+\frac{3x}{y}$. Ans. $\frac{2by+3x}{y}$.

4. Reduce $n-\frac{h}{3}$.

5. Reduce $1+\frac{2x^2-1}{5x}$.

6. Reduce $2y-\frac{3x}{y-2a}$.

Subtraction of Fractions.

155. The methods of performing subtraction in algebra, depend on the principle, that adding a negative quantity is equivalent to subtracting a positive one; and *v. v.* (Art. 74.) For the *subtraction of fractions*, then, we have the following simple *rule*.

Change the fraction to be subtracted, from positive to negative, or the contrary, and then proceed as in addition. (Art. 152.)

In making the required change, it will be expedient to alter, in some instances, the signs of the numerator, and in others, the sign before the dividing line, (Art. 147.) *so as to leave the latter always positive.*

Ex. 1. From $\frac{a}{b}$, subtract $\frac{h}{m}$.

First change $\frac{h}{m}$, the fraction to be subtracted, to $\frac{-h}{m}$.

Secondly, reduce the two fractions to a common denominator, making, $\frac{am}{bm}$ and $\frac{-bh}{bm}$.

Thirdly, the sum of the numerators $am-bh$, placed over the common denominator, gives the answer, $\frac{am-bh}{bm}$.

2. From $\frac{1-y}{2a}$ subtract $\frac{2}{x}$. Ans. $\frac{x-xy-4a}{2ax}$.

3. From $\frac{2n}{y}$ subtract $\frac{2a-1}{x}$. Ans. $\frac{2nx-2ay+y}{xy}$

4. From $\frac{2a-3}{5}$ subtract $\frac{a-5}{2}$. Ans. $\frac{19-a}{10}$.

5. From $\frac{1-x^2}{2y}$ subtract $-\frac{1-x}{3}$. Ans. $\frac{3+2y-2xy-3x^2}{6y}$.

6. From $\frac{1}{a}$ subtract $\frac{x}{2}$.

7. From $\frac{x+2}{y}$ subtract $\frac{y-2}{3x}$.

156. Fractions may also be subtracted, like integers, by setting them down, after their signs are changed, *without reducing them* to a common denominator.

From $\frac{h}{m}$ subtract $-\frac{h+d}{y}$. Ans. $\frac{h}{m}+\frac{h+d}{y}$.

In the same manner, an *integer* may be subtracted from a *fraction*, or a fraction from an integer.

From a subtract $\frac{b}{m}$. Ans. $a-\frac{b}{m}$.

157. Or the integer may be incorporated with the fraction, as in Art. 154.

Ex. 1. From $\frac{2}{x}$ subtract $2y$. Ans. $\frac{2}{x}-2y=\frac{2-2xy}{x}$.

2. From $3x-\frac{n}{2}$ subtract $-2x+\frac{3}{y}$. Ans. $\frac{10xy-ny-6}{2y}$.

3. From $\frac{a-2}{x}-1$ subtract $\frac{2-a}{x}$. Ans. $\frac{2a-4-x}{x}$.

4. From $2y+x-\frac{1-2a}{3}$, subtract $y-2x+\frac{a-2}{2}$.

Multiplication of Fractions.

158. By the definition of multiplication, multiplying by a fraction is taking a *part* of the multiplicand, as many times as there are like parts of an unit in the multiplier. (Art. 85.) Now the denominator of a fraction shows into what parts the integral unit is supposed to be divided; and the numerator shows how many of those parts belong to the given fraction. In multiplying by a fraction, therefore, the multiplicand is to be divided into such parts, as are denoted by the denominator; and then one of these parts is to be repeated, as many times, as is required by the numerator.

Suppose a is to be multiplied by $\frac{3}{4}$.

A fourth part of a is $\frac{a}{4}$.

This taken 3 times is $\frac{a}{4}+\frac{a}{4}+\frac{a}{4}=\frac{3a}{4}$. (Art. 152.)

Again, suppose $\frac{a}{b}$ is to be multiplied by $\frac{3}{4}$.

One fourth of $\frac{a}{b}$ is $\frac{a}{4b}$. (Art. 143.)

This taken 3 times is $\frac{a}{4b}+\frac{a}{4b}+\frac{a}{4b}=\frac{3a}{4b}$, the product required.

In a similar manner, any fractional multiplicand may be divided into parts, by multiplying the denominator; and one of the parts may be repeated, by multiplying the numerator. We have then the following *rule:*

To multiply fractions, multiply the numerators together, for a new numerator, and the denominators together, for a new denominator.

Ex. 1. Multiply $\frac{3b}{c}$ into $\frac{d}{2m}$. Product $\frac{3bd}{2cm}$.

2. Multiply $\frac{a+d}{y}$ into $\frac{4h}{m-2}$. Product $\frac{4ah+4dh}{my-2y}$.

3. Multiply $\frac{h(a+m)}{3}$ into $\frac{4}{(a-n)}$. Product $\frac{4h(a+m)}{3(a-n)}$.

4. Multiply $\frac{2-x}{1+y}$ into $\frac{a-b}{y-2}$.

5. Multiply $\frac{2}{3-n}$ into $\frac{1}{5}$.

159. The method of multiplying is the same, when there are *more than two* fractions to be multiplied together.

1. Multiply together $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{m}{y}$. Product $\frac{acm}{bdy}$.

For $\frac{a}{b}\times\frac{c}{d}$ is, by the last article $\frac{ac}{bd}$, and this into $\frac{m}{y}$ is $\frac{acm}{bdy}$.

2. Multiply $\frac{x}{y}$, $\frac{2y-1}{a}$, $\frac{n}{2h}$, $\frac{3}{2-b}$. Product $\frac{6nxy-3nx}{4ahy-2abhy}$.

3. Multiply $\frac{1}{3-n}$, $\frac{h}{2}$ and $\frac{n+2}{x}$.

4. Multiply $\frac{3x}{y^2}$, $\frac{5}{2}$ and $\frac{1+2b}{4-a}$.

160. The multiplication may sometimes be shortened, by *rejecting equal factors,* from the numerators and denominators.

1. Multiply $\frac{a}{r}$ into $\frac{h}{a}$ and $\frac{d}{y}$. Product $\frac{dh}{ry}$.

Here a, being in one of the numerators, and in one of the denominators, may be omitted. If it be retained, the product will be $\frac{adh}{ary}$. But this reduced to lower terms, by Art. 149, will become $\frac{dh}{ry}$ as before.

2. Multiply $\frac{2}{ax}$ into $\frac{ab}{3y}$ and $\frac{ay}{3}$. Product $\frac{2ab}{9x}$.

It is necessary that the factors rejected from the numerators be exactly equal to those which are rejected from the denominators. In the last example, a being in two of the numerators, and in only one of the denominators, must be retained in one of the numerators.

3. Multiply $\frac{a-b}{xy}$ into $\frac{hx}{3a}$. Product $\frac{ah-bh}{3ay}$.

Here, though the same letter a is in one of the numerators, and in one of the denominators, yet as it is not in *every term* of the numerator, it must not be cancelled.

4. Multiply $\frac{2xy-1}{a}$ into $\frac{n}{3x}$ and $\frac{2a}{y}$.

If any difficulty is found, in making these contractions, it will be better to perform the multiplication, without omitting any of the factors; and to reduce the product to lower terms afterwards.

161. When a *fraction* and an *integer* are multiplied together, *the* NUMERATOR *of the fraction is multiplied into the integer*. The denominator is not altered; except in cases where division of the denominator is substituted for multiplication of the numerator, according to Art. 144.

Thus $a\times\frac{m}{y}=\frac{am}{y}$. For $a=\frac{a}{1}$; and $\frac{a}{1}\times\frac{m}{y}=\frac{am}{y}$.

So $2n\times\frac{x}{d}\times\frac{h+1}{3}=\frac{2hnx+2nx}{3d}$. And $a\times\frac{1}{b}=\frac{a}{b}$.

The fraction may be *multiplied*, by *dividing its denominator* instead of multiplying the numerator; and this method is to be preferred, whenever the division can be actually performed.

Thus, to multiply $\frac{a}{2xy}$ by x, divide the denominator by x.

The result is $\frac{a}{2y}$.

162. ***A fraction is multiplied into a quantity equal to its denominator, by cancelling the denominator.***

Thus, to multiply the fraction $\frac{a}{b}$ by b, divide its denominator by b. The result is $\frac{a}{1}$ or a.

So the product of $\frac{3m}{a-y}$ into $a-y$ is $\frac{3m}{1}$ or $3m$.

And $\frac{h+3d}{3+m}\times(3+m)=h+3d$.

On the same principle, a fraction is multiplied into any *factor* in its denominator, by cancelling that factor.

Thus $\frac{a}{by}\times y$ is $\frac{a}{b}$.

For cancelling y in the denominator is dividing the denominator by y; and dividing the denominator is, in effect, multiplying the fraction.

So $\frac{h}{24}\times 6=\frac{h}{4}$. And $\frac{x}{ah+bh}\times(a+b)=\frac{x}{h}$.

163. From the definition of multiplication by a fraction, it follows that what is commonly called a *compound fraction*,* is the *product* of two or more fractions. Thus $\frac{3}{4}$ of $\frac{a}{b}$ is $\frac{3}{4}\times\frac{a}{b}$.

For $\frac{3}{4}$ of $\frac{a}{b}$, is $\frac{1}{4}$ of $\frac{a}{b}$ taken three times, that is, $\frac{a}{4b}+\frac{a}{4b}+\frac{a}{4b}$.

But this is the same as $\frac{a}{b}$ multiplied by $\frac{3}{4}$. (Art. 158.)

Hence, *reducing a compound fraction into a simple one, is the same as multiplying fractions into each other.*

Ex. 1. Reduce $\frac{2}{7}$ of $\frac{a}{b+2}$. Ans. $\frac{2a}{7b+14}$.

2. Reduce $\frac{1}{4}$ of $\frac{3}{5}$ of $\frac{a-x}{n-2}$. Ans. $\frac{3a-3x}{20n-40}$.

* By a compound fraction is meant a fraction *of* a fraction, and not a fraction whose numerator or denominator is a compound quantity.

3. Reduce $\frac{2}{3}$ of $\frac{5}{7}$ of $\frac{1}{2a-3b}$. Ans. $\frac{10}{42a-63b}$.

164. The expression $\frac{2}{3}a$, $\frac{1}{5}b$, $\frac{4}{7}y$, &c. are equivalent to $\frac{2a}{3}, \frac{b}{5}, \frac{4y}{7}$. For $\frac{2}{3}a$ is $\frac{2}{3}$ of a, which is equal to $\frac{2}{3}\times a=\frac{2a}{3}$. (Art. 161.) So $\frac{1}{5}b=\frac{1}{5}\times b=\frac{b}{5}$.

Division of Fractions.

165. *To divide one fraction by another, invert the divisor, and then proceed as in multiplication.* (Art. 158.)

Let it be required to divide $\frac{a}{b}$ by $\frac{c}{d}$. If we begin by dividing by c only, the quotient, according to Art. 143, will be $\frac{a}{bc}$. Then if the divisor, instead of being c, is c divided by d, the quotient must, according to Art. 137, be d times $\frac{a}{bc}$, that is, (Art. 161.) $\frac{ad}{bc}$; and this is the product of $\frac{a}{b}$ the dividend, into $\frac{d}{c}$, the divisor inverted. Hence we derive the rule stated above.

The division may be *proved*, by multiplying the divisor and quotient together. The product should be equal to the dividend. (Art. 113.)

Ex. 1. Divide $\frac{h^2}{3y}$ by $\frac{2h}{a}$. Ans. $\frac{h^2}{3y}\times\frac{a}{2h}=\frac{ah^2}{6hy}=\frac{ah}{6y}$.

Proof. $\frac{ah}{6y}\times\frac{2h}{a}=\frac{h^2}{3y}$.

2. Divide $\frac{2y-3}{x}$ by $\frac{a}{5y}$. Ans. $\frac{2y-3}{x}\times\frac{5y}{a}=\frac{10y^2-15y}{ax}$.

Proof. $\frac{10y^2-15y}{ax}\times\frac{a}{5y}=\frac{2y-3}{x}$.

3. Divide $\frac{an^2}{3y}$ by $\frac{nx}{3a}$. Ans. $\frac{an^2}{3y}\times\frac{3a}{nx}=\frac{a^2n}{xy}$.

Proof. $\frac{a^2n}{xy}\times\frac{nx}{3a}=\frac{an^2}{3y}$ the dividend.

4. Divide $\frac{20a^2}{3}$ by $\frac{5a}{3x}$. Ans. $\frac{20a^2}{3}\times\frac{3x}{5a}=4a^2x$.

5. Divide $\frac{2a^2-x}{2h}$ by $\frac{a^2-2x}{n}$.

6. Divide $\frac{ab+1}{3}$ by $\frac{2}{ab-1}$.

166. When a fraction is divided by an *integer*, the *denominator* of the fraction is multiplied into the integer.

Thus the quotient of $\frac{a}{b}$ divided by m, is $\frac{a}{bm}$.

For $m=\frac{m}{1}$; and by the last article, $\frac{a}{b}\div\frac{m}{1}=\frac{a}{b}\times\frac{1}{m}=\frac{a}{bm}$.

So $\frac{1}{a-b}\div h=\frac{1}{a-b}\times\frac{1}{h}=\frac{1}{ah-bh}$. And $\frac{3}{4}\div 6=\frac{3}{24}=\frac{1}{8}$.
See Art. 143.

And an integer is divided by a fraction, by multiplying the integer into the fraction inverted.

Thus the quotient of a divided by $\frac{b}{c}$, is $\frac{ac}{b}$.

For $a=\frac{a}{1}$; and $\frac{a}{1}\div\frac{b}{c}=\frac{a}{1}\times\frac{c}{b}=\frac{ac}{b}$.

In fractions, multiplication is made to perform the office of division; because division in the usual form often leaves a troublesome remainder: but there is no remainder in multiplication.

In many cases, there are methods of shortening the operation. But these will be suggested by practice, without the aid of particular rules.

167. By the definition, (Art. 43.) "the *reciprocal* of a quantity, is the quotient arising from dividing a unit by that quantity."

Therefore the reciprocal of $\frac{a}{b}$ is $1 \div \frac{a}{b} = 1 \times \frac{b}{a} = \frac{b}{a}$. That is,

The reciprocal of a fraction is the fraction inverted.

Thus the reciprocal of $\frac{b}{m+y}$ is $\frac{m+y}{b}$; the reciprocal of $\frac{1}{3y}$ is $\frac{3y}{1}$ or $3y$; the reciprocal of $\frac{1}{4}$ is 4.

Hence the reciprocal of a fraction whose numerator is 1, is the denominator of the fraction.

Thus the reciprocal of $\frac{1}{a}$ is a; of $\frac{1}{a+b}$, is $a+b$, &c.

168. A fraction sometimes occurs in the numerator or denominator of another fraction, as $\frac{\frac{2}{3}a}{b}$. It is often convenient, in the course of a calculation, to transfer such a fraction from the numerator to the denominator of the principal fraction, or the contrary. That this may be done without altering the value, if the fraction transferred be *inverted*, is evident from the following principles:

First, *Dividing* by a fraction, is the same as *multiplying* by the fraction *inverted*. (Art. 165 and 166.)

Secondly, *Dividing the numerator* of a fraction has the same effect on the value, as *multiplying the denominator*; and multiplying the numerator has the same effect, as dividing the denominator. (Art. 144.)

Thus in the expression $\frac{\frac{3}{5}a}{x}$ the numerator of $\frac{a}{x}$ is multiplied into $\frac{3}{5}$. But the value will be the same, if, instead of multiplying the numerator, we divide the denominator by $\frac{3}{5}$, that is, multiply the denominator by $\frac{5}{3}$.

Therefore $\frac{\frac{3}{5}a}{x} = \frac{a}{\frac{5}{3}x}$. So $\frac{h}{\frac{7}{9}m} = \frac{\frac{9}{7}h}{m}$

And $\frac{\frac{3}{4}d}{h+y} = \frac{d}{\frac{4}{3}(h+y)} = \frac{d}{\frac{4}{3}h + \frac{4}{3}y}$. And $\frac{a-x}{\frac{2}{7}m} = \frac{\frac{7}{2}a - \frac{7}{2}x}{m}$.

169. Multiplying the *numerator*, is in effect multiplying the *value* of the fraction. (Art. 142.) On this principle, a fraction may be cleared of a fractional co-efficient which occurs in its numerator.

Thus $\frac{\frac{3}{5}a}{b}=\frac{3}{5}\times\frac{a}{b}=\frac{3a}{5b}$. And $\frac{\frac{1}{5}a}{y}=\frac{1}{5}\times\frac{a}{y}=\frac{a}{5y}$.

And $\frac{\frac{1}{3}h+\frac{1}{3}x}{m}=\frac{1}{3}\times\frac{h+x}{m}=\frac{h+x}{3m}$. And $\frac{\frac{3}{4}x}{5a}=\frac{3x}{20a}$.

On the other hand, $\frac{3a}{7x}=\frac{3}{7}\times\frac{a}{x}=\frac{\frac{3}{7}a}{x}$.

And $\frac{a}{3y}=\frac{1}{3}\times\frac{a}{y}=\frac{\frac{1}{3}a}{y}$. And $\frac{4a}{5d+5x}=\frac{\frac{4}{5}a}{d+x}$.

170. But multiplying the *denominator*, by another fraction, is in effect *dividing* the value; (Art. 143.) that is, it is *multiplying* the value by the fraction *inverted*. The principal fraction may therefore be cleared of a fractional co-efficient which occurs in its denominator.

Thus $\frac{a}{\frac{3}{5}b}=\frac{a}{b}\div\frac{3}{5}=\frac{a}{b}\times\frac{5}{3}=\frac{5a}{3b}$. And $\frac{a}{\frac{2}{7}x}=\frac{7a}{2x}$.

And $\frac{a+h}{\frac{2}{9}y}=\frac{9a+9h}{2y}$. And $\frac{3h}{\frac{4}{7}m}=\frac{21h}{4m}$.

On the other hand, $\frac{7a}{3x}=\frac{a}{\frac{3}{7}x}$.

And $\frac{3y+3dx}{2m}=\frac{y+dx}{\frac{2}{3}m}$. And $\frac{3x}{y}=\frac{x}{\frac{1}{3}y}$.

171. The numerator or the denominator of a fraction, may be itself a fraction. The expression may be reduced to a more simple form, on the principles which have been applied in the preceding cases.

Thus $\frac{\frac{a}{b}}{\frac{c}{d}}=\frac{a}{b}\div\frac{c}{d}=\frac{ad}{bc}$. And $\frac{\frac{x}{y}}{h}=\frac{x}{hy}$. And $\frac{r}{\frac{m}{n}}=\frac{nr}{m}$.

SECTION VII.

SIMPLE EQUATIONS.

ART. **172.** The subjects of the preceding sections are introductory to what may be considered the peculiar province of algebra, the investigation of the values of unknown quantities by *equations*.

An EQUATION *is a proposition, expressing in algebraic characters, the equality between one quantity or set of quantities and another, or between different expressions for the same quantity.**

Thus $x+a=b+c$, is an equation, in which the sum of x and a, is equal to the sum of b and c. The quantities on the two sides of the sign of equality, are sometimes called the *members* of the equation; the several terms on the *left* constituting the *first* member, and those on the *right*, the *second* member.

173. An *identical* equation is one in which the two members are of the same form, or are such as may be reduced to the same form.

$$3x-ax+2=3x-ax+2, \quad (x-1)(x-2)=x^2-3x+2,$$

are equations of this kind.

An equation that is identical, will hold good for any values that may be substituted for the letters contained in it.

Thus, in the last equation, if x equal 10, each of the members will be 72; and if x equal 50, each member will be 2352.

But an equation which is not identical, will not be satisfied except for particular values of the letters.

Thus the equation $x^2=3x-2$ will be satisfied, if we substitute either 1 or 2 for x; but if we substitute 3, 4, or any other number, the equation will be destroyed.

174. An equation containing only the *first* power of an unknown quantity is called a *simple* equation, or an equation

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* See Note D.

of the *first degree*. One in which the highest power of the unknown quantity is a *square*, is called a *quadratic* equation, or an equation of the *second* degree. And in general, the *degree* of an equation is marked by the *highest power* of the unknown quantity contained in the equation.

Thus $ax=c$, $ax^2=bx+c$, $ax^3+bx^2=c$, $x^4+2x^3=x+a$, are equations of the *first, second, third* and *fourth* degrees; and are called also *simple, quadratic, cubic* and *biquadratic* equations.

It will here be convenient to attend only to *simple equations*, containing *one unknown quantity*.

175. The object aimed at, in what is called the *resolution* or *reduction* of an equation, is to *find the value of an unknown quantity*. In the first statement of the conditions of a problem, the known and unknown quantities are frequently thrown promiscuously together. To find the value of that which is required, it is necessary to bring it to stand by itself, while all the others are on the opposite side of the equation. But in doing this, care must be taken not to *destroy* the equation, by rendering the two members unequal. Many changes may be made in the arrangement of the terms, without affecting the equality of the sides.

176. *The reduction of an equation consists, then, in bringing the unknown quantity by itself, on one side, and all the known quantities on the other side, without destroying the equation.*

To effect this, it is evident that one of the members must be as much increased or diminished as the other. If a quantity be added to one, and not to the other, the equality will be destroyed.

But the members will remain equal;

If the same or equal quantities be *added* to each. Ax. 1.

If the same or equal quantities be *subtracted* from each. Ax. 2.

If each be *multiplied* by the same or equal quantities. Ax. 3.

If each be *divided* by the same or equal quantities. Ax. 4.

177. It may be farther observed that, in general, if the unknown quantity is connected with others by addition, multiplication, division, &c. the reduction is made by a *contrary* process. If a known quantity is *added* to the unknown, the equation is reduced by subtraction. If one is *multiplied* by

the other, the reduction is effected by *division*, &c. The reason of this will be seen, by attending to the several cases in the following articles.

The *known* quantities may be expressed either by letters or figures. The *unknown* quantity is represented by one of the last letters of the alphabet, generally, x, y, or z. (Art. 24.)

The principal reductions to be considered in this section, are those which are effected by *transposition*, *multiplication*, and *division*. These ought to be made perfectly familiar, as one or more of them will be necessary, in the resolution of almost every equation.

Transposition.

178. In the equation

$$x-7=9,$$

the number 7 being connected with the unknown quantity x by the sign $-$, the one is *subtracted* from the other. To reduce the equation by a contrary process, let 7 be *added* to both sides. It then becomes

$$x-7+7=9+7.$$

The equality of the members is preserved, because one is as much increased as the other. (Axiom 1.) But on one side, we have -7 and $+7$. As these are equal, and have contrary signs, they *balance each other*, and may be cancelled. (Art. 70.) The equation will then be

$$x=9+7.$$

Here the value of x is found. It is shown to be equal to $9+7$, that is to 16. The equation is therefore reduced. The unknown quantity is on one side by itself, and all the known quantities on the other side.

In the same manner, if	$x-b=a$
Adding b to both sides	$x-b+b=a+b$
And cancelling $(-b+b)$	$x=a+b.$

Here it will be seen that the last equation is the same as the first, except that b is on the opposite side, with a contrary sign.

Next suppose $\qquad y+c=d.$

Here c is *added* to the unknown quantity y. To reduce the equation by a contrary process, let c be subtracted from both sides, that is, let $-c$, be applied to both sides. We then have

$$y+c-c=d-c.$$

The equality of the members is not affected, because one is as much diminished as the other. When $(+c-c)$ is cancelled, the equation is reduced, and is

$$y=d-c.$$

This is the same as $y+c=d$, except that c has been transposed, and has received a contrary sign. We hence obtain the following *general rule:*

When known quantities are connected with the unknown quantity by the sign + or —, the equation is reduced by TRANSPOSING *the known quantities to the other side, and prefixing the contrary sign.*

This is called reducing an equation by *addition* or *subtraction*, because it is, in effect, adding or subtracting certain quantities, to or from each of the members.

Ex. 1. Reduce the equation $x+3b-m=h-d$
Transposing $+3b$, we have $x-m=h-d-3b$
And transposing $-m$, $x=h-d-3b+m.$

179. When several terms on the same side of an equation are *alike*, they may be united in one, by the rules for reduction in addition. (Art. 67 and 71.)

Ex. 2. Reduce the equation $x+5b-4h=7b$
Transposing $5b-4h$ $x=7b-5b+4h$
Uniting $7b-5b$ in one term $x=2b+4h.$

180. The *unknown* quantity must also be transposed, whenever it is on both sides of the equation. It is not material on which side it is finally placed. For if $x=3$, it is evident that $3=x$. It may be well, however, to bring it on that side, where it will have the affirmative sign, when the equation is reduced.

Ex. 3. Reduce the equation $2x+2h=h+d+3x$
By transposition $2h-h-d=3x-2x$
And $h-d=x$

181. When the *same term,* with the same sign, is on *opposite sides* of the equation, instead of transposing, we may *expunge* it from each. For this is only subtracting the same quantity from equal quantities. (Ax. 2.)

Ex. 4. Reduce the equation $x+3h+d=b+3h+7d$
Expunging $3h$ $x+d=b+7d$
And $x=b+6d.$

182. As *all* the terms of an equation may be transposed, or supposed to be transposed; and it is immaterial which member is written first; it is evident that the *signs of all the terms may be changed,* without affecting the equality.

Thus, if we have $x-b=d-a$
Then by transposition $-d+a=-x+b$
Or, inverting the members $-x+b=-d+a.$

183. If all the terms on *one* side of an equation be transposed, each member will be equal to 0.

Thus, if $x+b=d,$ then $x+b-d=0.$

It is frequently convenient to reduce an equation to this form, in which the positive and negative terms *balance* each other. In the example just given, $x+b$ is balanced by $-d.$ For in the first of the two equations, $x+b$ is equal to $d.$

Ex. 5. Reduce $3-2n+x=h-5+2x-2n.$
6. Reduce $3x-n^2+ah=n+2x-ah-n^2.$
7. Reduce $a-3x+12=b-12-4x-a^2.$
8. Reduce $14-y+ab+c^2=10-ab-c-2y.$

Reduction of Equations by Multiplication.

184. The unknown quantity, instead of being connected with a known quantity by the sign + or − , may be *divided* by it, as in the equation $\frac{x}{a}=b.$

Here the reduction can not be made, as in the preceding instances, by transposition. But if both members be *multiplied* by $a,$ (Art. 176.) the equation will become,

$$x=ab.$$

For a fraction is multiplied into its denominator, by removing the denominator. This has been proved from the properties of fractions. (Art. 162.)
Hence,

When the unknown quantity is DIVIDED *by a known quantity, the equation is reduced by* MULTIPLYING *each side by this known quantity.*

It must be observed, that *every* term of the equation is to be multiplied. For the several terms in each member constitute a compound multiplicand, which is to be multiplied according to Art. 95.

The same transpositions are to be made in this case, as in the preceding examples.

Ex. 1. Reduce the equation $\frac{x}{c}+a=b+d$

Multiplying both sides by c

The product is $x+ac=bc+cd$

And $x=bc+cd-ac.$

2. Reduce the equation $\frac{x-4}{6}+5=20.$

Ans. $x=94.$

3. Reduce the equation $\frac{x}{a+b}+d=h.$

Ans. $x=ah+bh-ad-bd.$

185. When the *unknown* quantity is in the *denominator* of a fraction, the reduction is made in a similar manner, by multiplying the equation by this denominator.

Ex. 4. Reduce the equation $\frac{6}{10-x}+7=8.$

Multiplying by $10-x$ $6+70-7x=80-8x.$

And $x=4.$

186. Though it is not generally *necessary*, yet it is often convenient, to remove the denominator from a fraction consisting of *known* quantities only. This may be done, in the same manner, as the denominator is removed from a fraction, which contains the unknown quantity.

Take for example $\frac{x}{a}=\frac{d}{b}+\frac{h}{c}$

Multiplying by a $x=\frac{ad}{b}+\frac{ah}{c}$

Multiplying by b $bx=ad+\frac{abh}{c}$

Multiplying by c $bcx=acd+abh.$

Or we may multiply by the product of *all* the denominators at once.

In the same equation $\frac{x}{a}=\frac{d}{b}+\frac{h}{c}$

Mutiplying by abc $\frac{abcx}{a}=\frac{abcd}{b}+\frac{abch}{c}.$

Then by cancelling from each term, the letter which is common to its numerator and denominator, (Art. 149.) we have $bcx=acd+abh$, as before.

Hence,

An equation may be cleared of FRACTIONS *by multiplying each side into all the* DENOMINATORS.

Thus the equation $\frac{x}{a}=\frac{b}{d}+\frac{e}{g}-\frac{h}{m}$

is the same as $dgmx=abgm+adem-adgh.$

And the equation $\frac{x}{2}=\frac{2}{3}+\frac{4}{5}+\frac{6}{2}$

is the same as $30x=40+48+180.$

187. An equation will be cleared of fractions, if each member be multiplied by *any common multiple* of the denominators, that is, by any quantity which is divisible by all the denominators. For the numerator of each fraction will be multiplied into this quantity, (Art. 161.) and will thus become divisible by the denominator.

The product of all the denominators is of course a multiple of them; but often it is not their least common multiple, and then it is convenient, in clearing the equation of frac-

tions, to multiply by the least common multiple of the denominators, instead of their product.

Thus, in the preceding example, it is sufficient to multiply by 30, instead of 60, which is the product of the denominators.

In clearing an equation of fractions, it will be necessary to observe, that the sign $-$ prefixed to any fraction, denotes that the whole value is to be subtracted, (Art. 147.) which is done by changing the signs of all the terms in the numerator.

The equation $$\frac{a-d}{x}=c-\frac{3b-2hm-6n}{r}$$

is the same as $$ar-dr=crx-3bx+2hmx+6nx.$$

Reduction of Equations by Division.

188. *When the unknown quantity is* MULTIPLIED *into any known quantity, the equation is reduced by* DIVIDING *both sides by this known quantity.* (Ax. 4.)

Ex. 1. Reduce the equation $\quad ax+b-3h=d$

By transposition $\quad ax=d+3h-b$

Dividing by a $\quad x=\dfrac{d+3h-b}{a}.$

2. Reduce the equation $\quad 2x=\dfrac{a}{c}-\dfrac{d}{h}+4b.$

Ans. $x=\dfrac{ah-cd+4bch}{2ch}.$

189. If the unknown quantity has co-efficients in *several terms*, the equation must be divided by *all* these co-efficients, connected by their signs, according to Art. 124.

Ex. 3. Reduce the equation $\quad 3x-bx=a-d$

That is, (Art. 119.) $\quad (3-b)x=a-d$

Dividing by $3-b$ $\quad x=\dfrac{a-d}{3-b}.$

4. Reduce the equation $\quad ax+x=h-4$

Dividing by $a+1$ $\quad x=\dfrac{h-4}{a+1}.$

5. Reduce the equation $$x-\frac{x-b}{h}=\frac{a+d}{4}$$

Ans. $x=\dfrac{ah+dh-4b}{4h-4}$.

190. If any quantity, either known or unknown, is found as a *factor* in *every term*, the equation may be *divided* by it. On the other hand, if any quantity is a *divisor* in every term, the equation may be *multiplied* by it. In this way, the factor or divisor will be removed, so as to render the expression more simple.

Ex. 6. Reduce the equation $ax+3ab=6ad+a$

Dividing by a $x+3b=6d+1$

And $x=6d+1-3b$.

7. Reduce the equation $$\frac{x+1}{x}-\frac{b}{x}=\frac{h-d}{x}$$

Ans. $x=h-d+b-1$.

8. Reduce the equation $x(a+b)-a-b=d(a+b)$

Dividing by $a+b$ (Art. 110 and 123.) $x-1=d$

And $x=d+1$.

191. If for any term or terms in an equation, any other expression of the same value be *substituted*, it is manifest that the equality of the sides will not be affected.

Thus, instead of 16, we may write 2×8, or $\frac{64}{4}$, or $25-9$, &c.

For these are only different forms of expression for the same quantity.

192. It will generally be well to have the several steps, in the reduction of equations, succeed each other in the following order.

First, Clear the equation of fractions. (Art. 186, 7.)

Secondly, Transpose and unite the terms. (Art. 178, &c.)

Thirdly, Divide by the co-efficients of the unknown quantity. (Arts. 188, 9.)

It is, however, sometimes convenient to unite terms that can be united, before clearing the equation of fractions.

Thus the following equation might be reduced, by first transposing 6 and uniting it with 7, and then proceeding in the usual way.

Examples.

1. Reduce the equation $\frac{3x}{4}+6=\frac{5x}{8}+7$

Clearing of fractions $24x+192=20x+224$
Transp. and uniting terms $4x=32$
Dividing by 4 $x=8$.

2. Reduce the equation $\frac{x}{a}+h=\frac{x}{b}-\frac{x}{c}+d$

Clearing of fractions $bcx+abx-acx=abcd-abch$

Dividing $x=\frac{abcd-abch}{bc+ab-ac}$.

3. Reduce $24+5x-30=159-6x$. Ans. $x=15$.

4. Reduce $\frac{2x-7}{3}-5=\frac{x}{4}-\frac{x-13}{2}$. Ans. $x=\frac{166}{11}$.

5. Reduce $3x+8+\frac{3}{2}x=7x-22$.

6. Reduce $5\frac{3}{4}-\frac{x}{2}=2x-16\frac{3}{4}$.

7. Reduce $\frac{3x}{5}-\frac{7x}{10}+\frac{3x}{4}=\frac{7x}{8}-15$.

8. Reduce $\frac{x}{2}+\frac{x}{3}-\frac{x}{5}=7x-\frac{x}{4}-734$.

9 Reduce $\frac{31}{6}x+134-\frac{13}{3}x=11x+\frac{3}{8}x+7\frac{1}{2}$.

10. Reduce $\frac{ax-10}{3}=\frac{x}{2}-\frac{b-5}{3}$. Ans. $x=\frac{30-2b}{2a-3}$

11. Reduce $\frac{x+3}{a}=\frac{x}{b}+12$. Ans. $x=\frac{3b(4a-1)}{b-a}$.

12. Reduce $\frac{a-1}{x}-1=b$.

13. Reduce $13-\frac{3}{2x-1}=11$.

14. Reduce $\frac{x}{2}-\frac{x}{3}=x-21$.

15. Reduce $\frac{x}{3}-\frac{x}{4}=\frac{x}{2}-\frac{5}{12}$.

16. Reduce $\frac{2x+5}{3}-12=\frac{x}{2}-\frac{x-7}{4}$.

17. Reduce $\frac{x+7}{2}-\frac{2x-7}{3}=\frac{x}{4}-15$.

18. Reduce $\frac{6x-1}{7}-\frac{3x-8}{5}=\frac{16-x}{2}-1+\frac{4x+1}{5}-x$.

19. Reduce $\frac{2x-3}{4}-\frac{4x-9}{5x-6}=\frac{x-1}{2}-\frac{7}{12}$.

20. Reduce $(a-x)(b+x)-b(a-c)+x^2=\frac{a^2c}{b}$.

Ans. $x=\frac{c(a+b)}{b}$

21. Reduce $\frac{7-x}{6}:5::\frac{5x-4}{3}:12$. See Art. 193.

193. Sometimes the conditions of a problem are at first stated, not in an equation, but by means of a *proportion*. To show how this may be reduced to an equation, it will be necessary to anticipate the subject of a future section, so far as to admit the principle that "when four quantities are in proportion, the product of the two extremes is equal to the product of the two means;" a principle which is at the foundation of the Rule of Three in arithmetic. See Arithmetic.

Thus, if $a:b::c:d$, then $ad=bc$.

And if $3:4::6:8$, then $3\times8=4\times6$. Hence,

A proportion is converted into an equation by making the product of the extremes, one side of the equation; and the product of the means, the other side.

Ex. 1. Reduce to an equation $ax : b :: ch : d.$
The product of the extremes is adx
The product of the means is bch
The equation is, therefore $adx = bch.$

2. Reduce to an equation $a+b : c :: h-m : y.$
The equation is $ay+by=ch-cm.$

194. *On the other hand, an equation may be converted into a proportion, by resolving one side of the equation into two factors, for the middle terms of the proportion; and the other side into two factors, for the extremes.*

As a quantity may often be resolved into different pairs of factors (Art. 30.); a variety of proportions may frequently be derived from the same equation.

Ex. 1. Reduce to a proportion $abc = deh.$
The side abc may be resolved into $a \times bc$, or $ab \times c$, or $ac \times b$.
And deh may be resolved into $d \times eh$, or $de \times h$, or $dh \times e$.

Therefore $a : d :: eh : bc$ And $ac : dh :: e : b$
And $ab : de :: h : c$ And $ac : d :: eh : b$, &c.

For in each of these instances, the product of the extremes is abc, and the product of the means deh.

2. Reduce to a proportion $ax+bx=cd-ch$
The first member may be resolved into $x(a+b)$
And the second into $c(d-h)$
Therefore $x : c :: d-h : a+b$ And $d-h : x :: a+b : c$, &c.

Solution of Problems.

195. In the solution of problems, by means of equations, two things are necessary: First, to translate the statement of the question from common to algebraic language, in such a manner as to form an equation: Secondly, to reduce this equation to a state in which the unknown quantity will stand by itself, and its value be given in known terms, on the opposite side. The manner in which the latter is effected, has already been considered. The former will probably occasion

more perplexity to a beginner; because the conditions of questions are so various in their nature, that the proper method of stating them cannot be easily learned, like the reduction of equations, by a system of definite rules. Practice will soon remove a great part of the difficulty.

The following rule may however be of some use.

Represent the unknown quantity by a letter, and indicate by algebraic signs, the operations that would be required to verify the value of the unknown quantity, if it were given.

196. It is one of the principal peculiarities of an algebraic solution, that the *quantity sought* is itself introduced into the operation. This enables us to make a statement of the conditions in the same form, as though the problem were already solved. Nothing then remains to be done, but to *reduce* the equation, and to find the aggregate value of the known quantities. (Art. 48.) As these are equal to the *unknown* quantity on the other side of the equation, the value of that also is determined, and therefore the problem is solved.

Problem 1. A man being asked how much he gave for his watch, replied: If you multiply the price by 4, and to the product add 70, and from this sum subtract 50, the remainder will be equal to 220 dollars.

To solve this, we must first translate the conditions of the problem, into such algebraic expressions as will form an equation.

The value of the unknown quantity will hereafter be found to be 50. If this value were given, we should verify it thus.

Let the price of the watch be 50.

This price is to be multiplied by 4, which makes 200

To the product, 70 is to be added, making $200+70$

From this, 50 is to be subtracted, making $200+70-50$

And the remainder is equal to 220, that is, $200+70-50=220$.

Reducing the first member, we see that this is a true equation, and hence infer that 50 is the true value of the unknown quantity.

According to the preceding rule then, we may form an equation for the solution of the problem, as follows.

Let the price of the watch be denoted by x.
Multiplying this by 4 makes $4x$
Adding 70 to the product makes $4x+70$
Subtracting 50 from this leaves $4x+70-50$.

Here we have a number of the conditions, expressed in algebraic terms; but have as yet no *equation*. We must observe then, that by the last condition of the problem, the preceding terms are said to be *equal* to 220.

We have, therefore, this equation $4x+70-50=220$
Which reduced gives $x=50$.

Here the value of x is found to be 50 dollars, which is the price of the watch.

197. We may *prove* whether we have obtained the true value of the letter which represents the unknown quantity, by substituting this value, for the letter itself, in the equation which contains the first statement of the conditions of the problem; and seeing whether the sides are equal, after the substitution is made. For if the answer thus satisfies the conditions proposed, it is the quantity sought. Thus, in the preceding example,

The original equation is $4x+70-50=220$
Substituting 50 for x, it becomes $4\times50+70-50=220$
That is, $220=220$.

Prob. 2. What number is that, to which if its half be added, and from the sum 20 be subtracted, the remainder will be a fourth of the number itself?

In stating questions of this kind, where fractions are concerned, it should be recollected, that $\frac{1}{3}x$ is the same as $\frac{x}{3}$; that $\frac{2}{5}x=\frac{2x}{5}$, &c. (Art. 164.)

In this problem, let x be put for the number required.

Then by the conditions proposed, $x+\frac{x}{2}-20=\frac{x}{4}$
And reducing the equation $x=16$.

Proof, $16+\frac{16}{2}-20=\frac{16}{4}$.

Prob. 3. A father divides his estate among his three sons, in such a manner, that,

The first has $1000 less than half of the whole;
The second has 800 less than one third of the whole;
The third has 600 less than one fourth of the whole.
What is the value of the estate?

If the whole estate be represented by x, then the several shares will be $\frac{x}{2}-1000$, and $\frac{x}{3}-800$, and $\frac{x}{4}-600$.

And as these constitute the whole estate, they are together equal to x.

We have then this equation $\frac{x}{2}-1000+\frac{x}{3}-800+\frac{x}{4}-600=x$.

Which reduced gives $x=28800$.

Proof, $\frac{28800}{2}-1000+\frac{28800}{3}-800+\frac{28800}{4}-600=28800$.

198. Letters may be employed to express the *known* quantities in an equation, as well as the unknown. A particular value is assigned to the letters, when they are introduced into the calculation; and at the close, the numbers are restored. (Art. 47.)

Prob. 4. If to a certain number 720 be added, and the sum be divided by 125; the quotient will be equal to 7392 divided by 462. What is that number?

Let $x=$ the number required.

$a=720$ $\qquad$ $d=7392$

$b=125$ $\qquad$ $h=462$.

Then by the conditions of the problem $\frac{x+a}{b}=\frac{d}{h}$

Therefore $x=\frac{bd-ah}{h}$

Restoring the numbers, $x=\frac{(125\times7392)-(720\times462)}{462}=1280$.

199. To avoid an unnecessary introduction of unknown quantities into an equation, it may be well to observe, in this place, that when the *sum* or *difference* of two quantities is given, both of them may be expressed by means of the same letter. For if one of the two quantities be subtracted from their sum, it is evident the remainder will be equal to the other. And if the difference of two quantities be subtracted from the greater, the remainder will be the less.

Thus if the sum of two numbers be $\quad 20$
And if one of them be represented by $\quad x$
The other will be equal to $\quad 20-x$.

Prob. 5. What two numbers are those whose sum is 50, and whose ratio is that of 3 to 2?

Let the larger number be denoted by x
Then the smaller will be $\quad 50-x$.
And $\quad x : 50-x :: 3 : 2$
Therefore $\quad x=30$, the greater
And $\quad 50-x=20$, the less.

Prob. 6. What two numbers are those whose sum is a, and whose ratio is that of m to n.

Let x be put for one number
Then the other will be $a-x$.
And $\quad x : a-x :: m : n$
Hence $\quad x=\dfrac{am}{m+n}$.

This is a general problem, of which the preceding is only a particular example, where the values of a, m and n are 50 3 and 2.

These letters may stand for any other numbers; and in each case, the formula $x=\dfrac{am}{m+n}$ will give the value of the unknown quantity: so that the solution of problem 6, affords the solution of an infinite number of particular problems similar to problem 5.

Prob. 7. What number is that, the double of which exceeds two thirds of its half by 45. Ans. 27.

Prob. 8. A farmer sold 14 bushels of wheat at a certain price; and afterward sold 22 bushels at the same rate, receiving 56 shillings more in this case than in the former. What was the price of a bushel? Ans. 7 shillings.

Prob. 9. What number is that the treble of which is as much above 25, as its half is below 38.

Prob. 10. A person bought 209 gallons of beer, which exactly filled four casks; the first held three times as much as the second, the second twice as much as the third, and the third twice as much as the fourth. How many gallons did each hold? Ans. 132, 44, 22, and 11 gallons.

Prob. 11. Divide 75 into two such parts, that the greater being divided by 5, and the less by 10, the difference of the quotients shall be 6.

Prob. 12. A silversmith has three pieces of metal, which together weigh 62 ounces. The second weighs 5 ounces more than the first, and the third 7 ounces more than the second. What are their weights?

Ans. 15, 20 and 27 ounces.

This problem may be generalized, as follows.

Prob. 13. The sum of three numbers is a; the second exceeds the first by m, and the third exceeds the second by n. What are the numbers?

Ans. $\frac{a-2m-n}{3}$, $\frac{a+m-n}{3}$, $\frac{a+m+2n}{3}$.

Prob. 14. What number is less by 6, than the sum of its half, its third and its fourth. Ans. 72.

Prob. 15. Two workmen received the same sum for their labor; but if one had received 12 dollars more, and the other 3 dollars less, the former would have had 4 times as much as the latter. What did they receive?

Ans. 8 dollars each.

Prob. 16. (The preceding problem generalized.) What number is that which, when increased by a, is m times as great as when diminished by b. Ans. $\frac{a+mb}{m-1}$.

Prob. 17. A fortress is garrisoned by 1800 men; and there are 7 times as many infantry, and 4 times as many artillery as cavalry. How many are there of each?

Ans. 1050 infantry, 600 artillery and 150 cavalry.

Prob. 18. Divide the number a into three such parts, that the first shall be m times, and the second n times, as great as the third.

$$\text{Ans. } \frac{ma}{m+n+1},\ \frac{na}{m+n+1},\ \frac{a}{m+n+1}.$$

Prob. 19. A farm of 660 acres is divided between three persons. C has as many acres as A and B together, and the portions of A and B are in the ratio of 4 to 7. How many acres has each?

Prob. 20. The ingredients of a loaf of bread are rice, flour and water; and the weight of the whole is 18 pounds. The weight of the rice increased by 2 pounds is one fifth of the weight of the flour, and the weight of the water is one eighth of the weight of the flour and rice together. What is the weight of each?

200. Though it is usual to begin the solution of a problem, by assuming a letter to represent some unknown quantity, it is sometimes best to adopt a different method.

Thus in the following example, where three numbers are required, which are as 2, 3 and 5, they may be represented by $2x$, $3x$ and $5x$; x being put for *half* of the first number. The value of x may then be found, and the required numbers obtained, by multiplying this value into 2, 3 and 5.

Prob. 21. There are three pieces of cloth, whose lengths are as 2, 3 and 5; and 12 yards being cut off from each, the whole quantity is diminished in the ratio of 10 to 7. What was the length of each piece at first?

Prob. 22. What two numbers are those, whose difference is 21, and whose sum is greater by 9 than three times the less?

Prob. 23. A company of men performed a piece of work in 6 days. If there had been 5 more, they would have finished the work in 4 days. Of how many did the company consist?

Prob. 24. In three towns, A, B and C, the whole number of inhabitants is 4600. The numbers in A and B are as 2 to 3, and in B and C as 5 to 7. What is the number in each of the towns?

Prob. 25. A certain sum is divided between three persons; A receives 1000 dollars less than the half, B ..00 dollars more

than the third, and C 300 dollars more than the fourth of the whole. How much does each receive?

Prob. 26. A trader first lost one third of his capital, and then gained 400 dollars; after which he lost one fourth of what he then had, and gained 200 dollars; lastly he lost one fifth of what he then had, and saved 1600 dollars. What had he at first?

Prob. 27. Divide 103 into four such parts, that, the first increased by 7, the second diminished by 5, the third multiplied by 4, and the fourth divided by 3, shall all be equal.

Prob. 28. A person has two pieces of land, A and B, the former of which is three fourths as large as the latter. Having sold 3 acres less than half of A, and 12 acres more than two thirds of B, he finds that the remaining part of B is only two fifths as large as that of A. What was the size of each field at first?

Prob. 29. A person has two kinds of sugar, one worth 12 cents, and the other 7 cents a pound. How much of each must he take, to make a mixture worth 10 cents a pound?

Prob. 30. A person has two kinds of sugar, one worth m cents a pound, and the other n cents. How much of each must he take to make a mixture worth r cents a pound?

Prob. 31. A reservoir containing 765 gallons of water, was emptied by two buckets in 36 minutes. The smaller bucket holding two thirds as much as the other, was emptied once a minute, and the larger one, three times in four minutes. What was the size of each?

Prob. 32. A man and his family consume a barrel of flour in 50 days. When he is absent, the barrel lasts 10 days longer. How long would it last, if he used it alone?

Prob. 33. A man and his family consume a barrel of flour in m days. When he is absent, the barrel lasts n days longer. How long would it last, if he used it alone?

Ans. $\frac{m(m+n)}{n}$ days.

Prob. 34. A merchant bought two kinds of wine in equal quantities; giving for one 4 shillings a gallon, and for the other 7. By mixing them and selling the mixture at 6 shillings a gallon, he gained 18 shillings. How many gallons of each did he buy?

Prob. 35. A merchant bought two kinds of wine in equal quantities, giving for one m shillings a gallon, and for the other n shillings a gallon. By mixing them and selling the mixture at r shillings a gallon, he gained a shillings. How many gallons of each did he buy? Ans. $\dfrac{a}{2r-m-n}$.

Prob. 36. What number is that, to which if 2, 5 and 11 be severally added, the first sum shall be to the second, as the second to the third?

Prob. 37. What number is that, to which if a, b and c be severally added, the first sum shall be to the second, as the second to the third? Ans. $\dfrac{b^2-ac}{a+c-2b}$.

Prob. 38. A, B and C make a joint stock. A puts in 300 dollars more than B, and 200 dollars less than C; and the sum of the shares of A and B, is to the sum of the shares of B and C as 7 to 9. What did each put in?

Prob. 39. A, B and C make a joint stock. A puts in m dollars more than B, and n dollars less than C; and the sum of the shares of A and B is to the sum of the shares of B and C, as p to q. What did each put in?

Ans. $\dfrac{p(n-m)+qm}{2(q-p)}$, $\dfrac{p(n+m)-qm}{2(q-p)}$, $\dfrac{q(2n+m)-p(n+m)}{2(q-p)}$.

Prob. 40. A laborer is paid 30 dollars for reaping 26 acres of wheat and rye, at the rate of a dollar an acre for the rye, and a dollar and a quarter for the wheat. How much would he have received for reaping the wheat alone?

Prob. 41. Divide 15 into two such parts that the difference of their squares shall be 105.

Prob. 42. Divide the number a into two such parts that the difference of their squares shall be d.

Ans. $\dfrac{a^2+d}{2a}$, $\dfrac{a^2-d}{2a}$.

Prob. 43. Out of a certain sum of money a man paid his creditors 432 dollars; one third of the remainder he lent his friend; he then spent one fourth of what still remained; after which he had one fifth of the money left? How much had he at first?

Prob. 44. A person being asked the hour, answered that it was between two and three, and the hour and minute hands were together. What was the time?

Prob. 45. A merchant buys a cask of wine for 150 dollars, and sells 5 gallons less than two thirds of the whole at a profit of 50 per cent. He afterwards sells the remainder at a profit of 125 per cent. and finds that he has cleared 80 per cent. by the whole transaction. How many gallons does the cask contain?

Prob. 46. In a naval engagement the number of ships taken was one fifth of the whole; the number burnt was 1 less, and the number sunk 2 more than the number taken; and 13 escaped. Of how many did the fleet consist?

Prob. 47. The yearly rent of a farm is 80 dollars in money, and a certain quantity of wheat. When wheat is worth a dollar a bushel, the rent is 8 dollars per acre; but when it is worth a dollar and a quarter, the rent per acre is 9 dollars. Of how many acres does the farm consist?

Prob. 48. The yearly rent of a farm is a dollars in money, and a certain quantity of wheat. When wheat is worth m dollars a bushel, the rent is p dollars per acre; but when it is worth n dollars, the rent per acre is q dollars. Of how many acres does the farm consist? Ans. $\frac{(n-m)a}{np-mq}$.

Prob. 49. A and B are traveling in the same direction, and A is 15 miles in advance of B. If B goes five miles while A goes four, how far must he travel to overtake A?

Prob. 50. A and B are traveling in the same direction, and A is d miles in advance of B. If B goes m miles while A goes n, how far must he travel to overtake A?

Ans. $\frac{md}{m-n}$.

SECTION VIII.

SIMPLE EQUATIONS CONTAINING TWO OR MORE UNKNOWN QUANTITIES.

ART. **201.** IN the examples which have been given of the resolution of equations, in the preceding sections, each problem has contained only *one* unknown quantity. Or if, in some instances, there have been *two*, they have been so related to each other, that they have both been expressed by means of the same letter. (Art. 199.)

But cases frequently occur, in which *several* unknown quantities are introduced into the same calculation. And if the problem is of such a nature as to admit of a determinate answer, there will arise from the conditions, as many equations independent of each other, as there are unknown quantities.

Equations are said to be *independent*, when they express different conditions; and *dependent*, when they express the same conditions under different forms. The former are not convertible into each other. But the latter may be changed from one form to the other, by the methods of reduction which have been considered. Thus $b-x=y$, and $b=y+x$, are dependent equations, because one is formed from the other by merely transposing x.

In solving a problem, it is necessary first to find the value of one of the unknown quantities, and then of the others in succession. To do this, we must derive from the equations which are given, a new equation, from which all the unknown quantities except one shall be excluded.

SOLUTION OF PROBLEMS WHICH CONTAIN TWO UNKNOWN QUANTITIES.

202. There are *three* methods employed in *eliminating* unknown quantities, called

Elimination *by comparison*,

Elimination *by substitution*, and

Elimination *by addition and subtraction.*

Elimination by Comparison.

203. Suppose the following equations are given.

1. $x+y=14$
2. $x-y=2.$

If y be transposed in each, they will become

1. $x=14-y$
2. $x=2\ +y.$

Here the first member of each of the equations is x, and the second member of each is *equal* to x. But according to axiom 11th, quantities which are respectively equal to any other quantity are equal to each other; therefore,

$$2+y=14-y.$$

Here we have a new equation, which contains only the unknown quantity y. Hence,

Rule I. To eliminate one of two unknown quantities, and deduce one equation from two;

Find the value of one of the unknown quantities in each of the equations, and form a new equation by making one of these values equal to the other.

That quantity which is the least involved should be the one which is chosen to be eliminated.

For the convenience of referring to different parts of a solution, the several steps will, in future be numbered. When an equation is formed from one *immediately preceding*, it will be unnecessary to specify it. In other cases, the number of the equation or equations from which a new one is derived, will be referred to.

Prob. 1. To find two numbers such, that

Their sum shall be 24; and

The greater shall be equal to five times the less.

Let x=the greater; And y=the less.

1. By the first condition, $\left.\begin{matrix} x+y=24 \\ x=5y \end{matrix}\right\}$
2. By the second,
3. Transp. y in the first equation, $x=24-y$
4. Making the 2d and 3d equal, $5y=24-y$
5. And $y=4$, the less number.

Prob. 2. To find two numbers such that

Their difference shall be a; and

The greater shall be equal to m times the less.

Ans. $\frac{ma}{m-1}$, $\frac{a}{m-1}$.

Prob. 3. Given $\left.\begin{array}{l} ax-by=c \\ x-\ y=d \end{array}\right\}$ To find x. Ans. $x=\frac{bd-c}{b-a}$.

Elimination by Substitution.

204. Suppose the equations given are, as in the former case,

$$1.\quad x+y=14$$
$$2.\quad x-y=2.$$

If y be transposed, the first equation will become,

$$x=14-y.$$

And as x is here equal to $14-y$, we may in the second equation *substitute* this value of x instead of x itself. The second equation will then be converted into

$$14-2y=2.$$

The equality of the two sides is not affected by this alteration, because we only change one quantity x for another which is equal to it. By this means we obtain an equation which contains only one unknown quantity. Hence,

Rule II. To eliminate an unknown quantity,

Find the value of one of the unknown quantities, in one of the equations; and then in the other equation SUBSTITUTE *this value for the unknown quantity itself.*

Prob. 4. A privateer in chase of a ship 20 miles distant, sails 8 miles, while the ship sails 7. How far must the privateer sail before she overtakes the ship?

It is evident that the whole distance which the privateer sails during the chase, must be to the distance which the ship sails in the same time, as 8 to 7.

Let $x=$ the distance which the privateer sails,
And $y=$ the distance which the ship sails.

1. By the supposition, $\left.\begin{array}{r}x=y+20\\ x:y::8:7\end{array}\right\}$
2. And also,
3. Art. 193, $y=\frac{7}{8}x$
4. Substituting $\frac{7}{8}x$ for y, in the 1st equation, $x=\frac{7}{8}x+20$
5. Therefore, $x=160$.

Prob. 5. What fraction is that, which becomes two thirds, when 2 is added to its numerator, and becomes one half, when 1 is added to its denominator.

Ans. $\frac{8}{15}$.

Prob. 6. There are two numbers,
Whose sum is to their difference as 9 to 4; and
The greater exceeds twice the less by 3.
What is the less number? Ans. 5.

Elimination by Addition and Subtraction.

205. Suppose the given equations are

$$\left.\begin{array}{r}x+3y=a\\ 2x-3y=b\end{array}\right\}$$

If we *add together* the first members of these two equations, and also the second members, we shall have

$$3x=a+b,$$

an equation which contains only the unknown quantity x. The other, having equal co-efficients with contrary signs, has disappeared. (Art. 70.) The equality of the sides is preserved, because we have only added equal quantities to equal quantities.

Again, suppose $3x+y=h$
And $2x+y=d$ $\Big\}$

If we *subtract* the last equation from the first, we shall have

$$x=h-d$$

where y is eliminated, without affecting the equality of the sides.

Again, suppose $\left.\begin{array}{r}3x-2y=a \\ x+4y=b\end{array}\right\}$
And

Multiplying the 1st by 2, $6x-4y=2a$

Then adding the 2d and 3d, $7x=b+2a$.

Again, suppose $\left.\begin{array}{r}2x-4y=10 \\ x+3y=35\end{array}\right\}$
And

Dividing the 1st by 2, $x-2y=5$

Then subtracting the 3d from the 2d, $5y=30$. Hence,

Rule III. To eliminate an unknown quantity,

MULTIPLY *or* DIVIDE *the equations, if necessary, in such a manner that the term which contains one of the unknown quantities shall be the same in both.*

Then SUBTRACT *one equation from the other, if the signs of this unknown quantity are* ALIKE, *or* ADD *them together, if the signs are* UNLIKE.

It must be kept in mind that both members of an equation are always to be increased or diminished, multiplied or divided alike. (Art. 176.)

Prob. 7. The numbers in two opposing armies are such, that,

The sum of both is 21110; and

Twice the number in the greater army, added to three times the number in the less, is 52219.

What is the number in the greater army?

Let $x=$ the greater. And $y=$ the less.

1. By the first condition, $\left.\begin{array}{r}x+y=21110 \\ 2x+3y=52219\end{array}\right\}$
2. By the second,
3. Multiplying the 1st by 3, $3x+3y=63330$
4. Subtracting the 2d from the 3d, $x=11111$.

Prob. 8. Given $2x+3y=7$, and $3x-2y=4$, to find the value of x.

Multiply the first equation by 2, the second by 3, and add the results. Ans. $x=2$.

Prob. 9. Given $12x-3y=30$, and $2x+5y=16$, to find the value of y.

Divide the first equation by 3, multiply the second by 2, and then subtract one from the other. Ans. $y=2$.

In the succeeding problems, either of the three rules for eliminating unknown quantities will be made use of, as will in each case be most convenient.

206. When *one* of the unknown quantities is determined, the other may be easily obtained, by going back to an equation which contains both, and substituting instead of that which is already found, its numerical value.

Prob. 10. The mast of a ship consists of two parts;

One third of the lower part added to one sixth of the upper part, is equal to 28; and,

Five times the lower part, diminished by six times the upper part, is equal to 12.

What is the height of the mast?

Let $x=$ the lower part; And $y=$ the upper part.

1. By the first condition, $\left.\begin{array}{l}\frac{1}{3}x+\frac{1}{6}y=28\\ 5x-6y=12\end{array}\right\}$
2. By the second,
3. Multiplying the 1st by 6, $2x+y=168$
4. Dividing the 2d by 6, $\frac{5}{6}x-y=2$
5. Adding the 3d and 4th, $2x+\frac{5}{6}x=170$
6. Therefore, $x=60$, the lower part.

Then by the 3d step, $2x+y=168$

That is, substituting 60 for x, $120+y=168$

Hence, $y=168-120=48$, the upper part.

Prob. 11. Given $\left.\begin{array}{l}\dfrac{x}{5}+\dfrac{y}{6}=2\\[2ex] \text{And} \quad \dfrac{x}{3}+\dfrac{y}{4}=\dfrac{19}{6}\end{array}\right\}$ To find x and y.

Ans. $x=5$, $y=6$.

Prob. 12. Given $(x+1)(y+2)=(x-1)(y-2)+26$

And $5x-8y=1$

To find the values of x and y.

Ans. $x=5$, $y=3$.

Prob. 13. Given $\frac{x-4}{3}+\frac{y-x}{2}+\frac{10-y}{12}=\frac{19}{12}$

And $\frac{x+y}{6}+\frac{x+1}{2}-\frac{y-1}{3}=3$

To find the values of x and y.

Ans. $x=5$, $y=7$.

Prob. 14. To find x and y from

1. The equation $\frac{8}{x}\ \frac{5}{y}=\frac{1}{6}$

2. And $\frac{7}{x}-\frac{3}{y}=\frac{5}{6}$

3. Multiplying the 1st by 3, $\frac{24}{x}-\frac{15}{y}=\frac{3}{6}$

4. Multiplying the 2d by 5, $\frac{35}{x}-\frac{15}{y}=\frac{25}{6}$

5. Subtracting the 3d from the 4th, $\frac{11}{x}=\frac{22}{6}$

6. Therefore $x=3$

7. Substituting 3 for x in the 2d, $\frac{7}{3}-\frac{3}{y}=\frac{5}{6}$

8. Hence $y=2$.

Prob. 15. Given $\frac{a}{x}-\frac{b}{y}=m$

And $\frac{c}{x}-\frac{d}{y}=n$

To find the values of x and y.

Ans. $x=\frac{bc-ad}{bn-dm}$, $y=\frac{bc-ad}{an-cm}$.

Prob. 16. If a certain number, consisting of two digits, be divided by the left hand digit increased by 1, the quotient will be 9; but if it be divided by the right hand digit diminished by 1, the quotient will be $\frac{9}{2}$. What is the number?

Let $x=$ the left hand digit, and $y=$ the right hand digit.

As the local value of figures increases in a tenfold ratio from right to left, the required number $=10x+y$.

Then $$\frac{10x+y}{x+1}=9$$

And $$\frac{10x+y}{y-1}=\frac{9}{2}.$$

The number sought is, therefore, 27.

Prob. 17. If a certain number consisting of two digits, be divided by the left hand digit increased by m, the quotient will be a; but if it be divided by the right hand digit diminished by n, the quotient will be b. What is the number?

Ans. $\frac{ab(10m-n)}{a+10b-ab}$.

Prob. 18. To find a fraction such that,

If the numerator be doubled, and the denominator be diminished by 2, the fraction will be equal to $\frac{7}{5}$; but

If the denominator be doubled, and the numerator be diminished by 3, the fraction will be equal to $\frac{1}{6}$.

Ans. $\frac{7}{12}$.

Prob. 19. What two numbers are those, whose sum divided by the greater is equal to $\frac{9}{5}$, and whose difference is less than half the greater by 6? Ans. 20 and 16.

Prob. 20. What two numbers are those, whose sum divided by the greater is equal to a, and whose difference is less than half the greater by b? Ans. $\frac{2b}{2a-3}$, $\frac{2b(a-1)}{2a-3}$.

Prob. 21. If B gives to A 5 dollars, A will then have half as much money as B. But if A gives to B ten dollars, B will then have five times as much as A. How many dollars has each? Ans. A has 25 and B 65 dollars.

Prob. 22. A merchant mixes wheat flour which cost him 5 dollars a barrel, with rye flour which cost him 3 dollars a barrel, in such proportion as to gain $33\frac{1}{3}$ per cent, by selling the mixture at 6 dollars a barrel. How much wheat is there in 40 barrels of the mixture? Ans. 30 barrels.

Prob. 23. A merchant mixes wheat flour, which cost him m dollars a barrel, with rye flour which cost him n dollars a barrel, in such proportion as to gain p per cent, by selling the mixture at r dollars a barrel. How much wheat is there in a barrels of the mixture?

Ans. $\frac{(100r-100n-pn)a}{(m-n)(100+p)}$.

Prob. 24. A and B engage to reap a field of wheat in 12 days. But after working 4 days, A is called off, and B is left to finish the work, which he does in 20 days more. In what time would A alone reap the field? Ans. 20 days.

Prob. 25. A and B engage to reap a field of wheat in m days. But after working n days, A is called off, and B is left to finish the work, which he does in r days more. In how many days would A alone reap the field?

Ans. $\frac{mr}{r+n-m}$.

Solution of Problems which contain Three or more Unknown Quantities.

207. In the examples hitherto given, each has contained no more than two unknown quantities. And two independent equations have been sufficient to express the conditions of the question. But problems may involve three or more unknown quantities; and may require for their solution as many independent equations.

$$\left.\begin{array}{ll}\text{Suppose} & x+\ y+\ z=12 \\ \text{And} & x+2y-2z=10 \\ \text{And} & x+\ y-\ z=4\end{array}\right\}\text{are given, to find } x,\ y \text{ and } z.$$

From these three equations, two others may be derived, which shall contain only two unknown quantities. One of the three in the original equations may be eliminated, in the same manner as when there are, at first, only two, by the rules in Arts. 203, 4, 5.

In the equations given above, if we transpose y and z, we shall have,

$$\left.\begin{array}{ll}\text{In the first,} & x=12-\ y-\ z \\ \text{In the second,} & x=10-2y+2z \\ \text{In the third,} & x=\ 4-\ y+\ z\end{array}\right\}$$

From these we may deduce two new equations, from which x shall be excluded.

By making the 1st and 2d equal, $12-y-z=10-2y+2z$
By making the 2d and 3d equal, $10-2y+2z=4-y+z$
Reducing the first of these two, $y=3z-2$
Reducing the second, $y=z+6$

From these two equations, one may be derived containing only one unknown quantity.

Making one equal to the other, $3z-2=z+6$
And $z=4$. Hence,

To solve a problem containing *three* unknown quantities, and producing three independent equations,

First, from the three equations deduce two, containing only two unknown quantities.

Then, from these two deduce one, containing only one unknown quantity.

For making these reductions, the rules already given are sufficient. (Art. 203, 4, 5.)

Prob. 26. Let there be given,

$$\left.\begin{array}{lll} 1. & \text{The equation} & x+5y+6z=53 \\ 2. & \text{And} & x+3y+3z=30 \\ 3. & \text{And} & x+y+z=12 \end{array}\right\} \text{To find } x, y \text{ and } z.$$

From these three equations to derive two, containing only two unknown quantities,

4. Subtract the 2d from the 1st, $2y+3z=23$
5. Subtract the 3d from the 2d, $2y+2z=18$

From these two, to derive one,

6. Subtract the 5th from the 4th, $z=5$.

To find x and y, we have only to take their values from the third and fifth equations. (Art. 206.)

7. Reducing the 5th, $y=9-z=9-5=4$.
8. Transposing in the 3d, $x=12-z-y=12-5-4=3$.

Prob. 27. To find x, y and z, from

1. The equation $\quad x+y+z=12$
2. And $\quad x+2y+3z=20$
3. And $\quad \frac{1}{2}x+\frac{1}{3}y+z=6$
4. Multiplying the 1st by 3, $\quad 3x+3y+3z=36$
5. Subtracting the 2d from the 4th, $\quad 2x+y=16$
6. Subtracting the 3d from the 1st, $\quad x-\frac{1}{2}x+y-\frac{1}{3}y=6$
7. Clearing the 6th of fractions, $\quad 4x+3y=36$
8. Multiplying the 5th by 3, $\quad 6x+3y=48$
9. Subtracting the 7th from the 8th, $2x=12$. And $x=6$.
10. Reducing the 7th, $\quad y=\dfrac{36-4x}{3}=\dfrac{36-24}{3}=4.$
11. Reducing the 1st, $\quad z=12-x-y=12-6-4=2.$

In this example, all the reductions have been made according to the *third* rule for eliminating unknown quantities. (Art. 205.) But either of the three may be used at pleasure.

208. A calculation may often be very much abridged, by the exercise of judgment in stating the question, in selecting the equations from which others are to be deduced, in simplifying fractional expressions, &c. The skill which is necessary for this purpose, however, is to be acquired, not from a system of rules, but from practice, and a habit of attention to the peculiarities in the conditions of different problems, the variety of ways in which the same quantity may be expressed, the numerous forms which equations may assume, &c. In many of the examples in this and the preceding section, the processes might have been shortened. But the object has been to illustrate general principles rather than to furnish specimens of expeditious solutions. The learner will do well, as he passes along, to exercise his skill in abridging the calculations which are given, or substituting others in their stead.

Prob. 28. Given $\left\{\begin{array}{l} x+y-z=a \\ x+z-y=b \\ y+z-x=c \end{array}\right\}$ To find x, y and z.

Ans. $x=\dfrac{a+b}{2}$, $y=\dfrac{a+c}{2}$, $z=\dfrac{b+c}{2}$.

Prob. 29. To divide 46 into three such parts, that if 10 be added to the first and second, they shall be in the ratio of 5 to 9, and if 3 be taken from the second and third, they shall be in the ratio of 2 to 3. Ans. 5, 17 and 24.

209. The learner must exercise his own judgment, as to the choice of the quantity to be first eliminated. It will generally be best to begin with that which is most free from coefficients, fractions, &c.

Prob. 30. A says to B, give me 10 dollars and I shall have twice as much money as you; B says to C, give me 20 dollars and I shall have thrice as much as you; C says to A, givé me 6 dollars, and I shall have five times as much as you. How much has each?

Ans. A $14, B $22, C $34.

Prob. 31. To find x, y and z, from

$$\left.\begin{array}{ll}\text{The equation} & x+\frac{1}{2}y+\frac{1}{3}z=18 \\ \text{And} & y+\frac{1}{3}z+\frac{1}{4}x=23 \\ \text{And} & z+\frac{1}{2}x+\frac{1}{3}y=25\end{array}\right\}$$

Ans. $x=6$, $y=12$, $z=18$.

Prob. 32. Given $\left\{\begin{array}{l}\frac{1}{x}+\frac{1}{y}=a \\ \frac{1}{x}+\frac{1}{z}=b \\ \frac{1}{y}+\frac{1}{z}=c\end{array}\right\}$ To find x, y and z.

Ans. $x=\frac{2}{a+b-c}$, $y=\frac{2}{a+c-b}$, $z=\frac{2}{b+c-a}$

Prob. 33. In a number consisting of three digits, the middle digit is half the sum of the other two. If the number be divided by the sum of its digits, the quotient will be 26; and if 198 be added to the number, the order of the digits will be reversed. What is the number? Ans. 234.

210. The same method which is employed for the reduction of three equations, may be extended to four, five, or any number of equations, containing as many unknown quantities.

The unknown quantities may be eliminated, one after another, and the number of equations may be reduced by successive steps from five to four, from four to three, from three to two, &c.

The *general rule* for the reduction of n equations containing n unknown quantities may be stated thus.

Combine any one of the equations with each of the others, so as to eliminate in each case the same unknown quantity. There will then be $n-1$ new equations, containing $n-1$ unknown quantities.

Eliminate another unknown quantity by combining one of these new equations with each of the others. This will give $n-2$ equations, containing $n-2$ unknown quantities.

Continue this process till there is obtained a single equation containing one unknown quantity. From this equation deduce the value of this unknown quantity; and then by going back to preceding equations, determine successively the values of the other unknown quantities.

Prob. 34. To find w, x, y and z from

$$\left.\begin{array}{lr}
\text{1. The equation} & x+2y-z+w=6 \\
\text{2. And} & x-y+3z-w=4 \\
\text{3. And} & 2w+y+2z-x=15 \\
\text{4. And} & x+y+z+2w=14
\end{array}\right\}$$

Combining the first equation with each of the other three,

$$\begin{array}{lr}
\text{5.} & 3y-4z+2w=2 \\
\text{6.} & 3y+z+3w=21 \\
\text{7.} & y-2z-w=-8
\end{array}$$

Eliminating y from these new equations,

$$\begin{array}{lr}
\text{8.} & 5z+w=19 \\
\text{9.} & 2z+5w=26 \\
\text{Hence} & 23z=69, \text{ and } z=3.
\end{array}$$

By going back to the 8th or 9th equation, and substituting for z its value, we find $w=4$. The value of y is next obtained from one of the equations 5, 6, and 7; and finally the value of x is deduced from one of the original equations. These values are 2 and 1.

211. The method of combining the equations may be somewhat varied to suit different cases. Instead of combining one equation with several others, as the rule directs, we may arrange the equations in any order, and combine the second with the first, the third with either of the two first, the fourth with either of the three first, and so on to the last.

Prob. 35. To find u, x, y and z, from the equations

1. $9u-4x+10y+3z=9$
2. $3x+6y-2z+9u=6$
3. $4y+5z-6u-5x=5$
4. $2z-3u-3x+8y=3$

Ans. $u=\frac{1}{3}$, $x=2$, $y=\frac{1}{2}$, $z=3$.

Prob. 36. Divide 18 into four such parts, that the first with half the sum of the other three shall be 10, the second with a third of the sum of the other three shall be 8, and the third with twice the sum of the other three shall be 31.

Ans. The parts are 2, 3, 5 and 8.

Prob. 37. A number consists of four digits whose sum is 19. The first or left-hand digit is equal to the sum of the second and third; the second is equal to the sum of the third and fourth; and if 8082 be taken from the number, the order of the digits will be inverted. What is the number?

Ans. 9541.

212. If in the algebraic statement of the conditions of a problem, the original equations are more numerous than the unknown quantities; these equations will either be *contradictory*, or one or more of them will be *superfluous*.

Thus the equations $\left\{\begin{matrix} 3x=60 \\ \frac{1}{2}x=20 \end{matrix}\right\}$ are contradictory.

For by the first $x=20$, while by the second $x=40$.

But if the latter be altered, so as to give to x the same value as the former, it will be useless, in the statement of a problem. For nothing can be determined from the one, which cannot be from the other.

Thus of the equations $\left\{\begin{matrix} 3x=60 \\ \frac{1}{2}x=10 \end{matrix}\right\}$ one is superfluous.

For either of them is sufficient to determine the value of x. They are not *independent* equations. (Art. 201.) One is convertible into the other. For if we divide the 1st by 6, it will become the same as the second. Or if we multiply the second by 6, it will become the same as the first.

213. But if the number of independent equations produced from the conditions of a question, is *less* than the number of unknown quantities, the question is not sufficiently limited to admit of a definite answer. For each equation can limit but one quantity. And to enable us to find this quantity, all the others connected with it, must either be previously known, or be determined from other equations. If this is not the case, there will be a variety of answers which will equally satisfy the conditions of the question. If, for instance, x and y are required from the equation

$$x+y=100,$$

there may be fifty different answers. The values of x and y may be either 99 and 1, or 98 and 2, or 97 and 3, &c. For the sum of each of these pairs of numbers is equal to 100. But if there is a second equation which is independent of the former, as $x-y=96$, then x and y are determined; their values are 98 and 2. No other values will satisfy both equations.

214. For the sake of abridging the solution of a problem, however, the number of independent equations actually put upon paper is frequently less than the number of unknown quantities.

Suppose we are required to divide 100 into two such parts, that the greater shall be equal to three times the less. If we put x for the greater, the less will be $100-x$. (Art. 19?.)

Then by the supposition,	$x=300-3x$.
Tranposing and dividing,	$x=75$, the greater.
And	$100-75=25$, the less.

Here, two unknown quantities are found, although there appears to be but one independent equation. The reason of this is, that a part of the solution has been omitted, because it is so simple, as to be easily supplied by the mind. To have a view of the whole, without abridging, let $x=$ the greater number, and $y=$ the less.

1. Then by the supposition, $\left.\begin{array}{l} x+y=100 \\ 3y=x \end{array}\right\}$
2. And
3. Transposing x in the 1st, $y=100-x$
4. Dividing the 2d by 3, $y=\frac{1}{3}x$
5. Making the 3d and 4th equal, $\frac{1}{3}x=100-x$
6. Multiplying by 3, $x=300-3x$
7. Transposing and dividing, $x=75$, the greater.
8. By the 3d step, $y=100-x=25$, the less.

By comparing these two solutions with each other, it will be seen that the first begins at the 6th step of the latter, all the preceding parts being omitted, because they are too simple to require the formality of writing down.

215. In most cases also, the solution of a problem which contains many unknown quantities, may be abridged, by particular artifices in *substituting* a single letter for several.

Suppose four numbers, u, x, y and z, are required, of which

The sum of the three first is	13
The sum of the two first and last	17
The sum of the first and two last	18
The sum of the three last	21

Then 1. $u+x+y=13$

2. $u+x+z=17$
3. $u+y+z=18$
4. $x+y+z=21.$

Let s be substituted for the *sum* of the four numbers, that is, for $u+x+y+z$. It will be seen that of these four equations,

The first contains all the letters except z, that is, $s-z=13$
The second contains all except y, that is, $s-y=17$
The third contains all except x, that is, $s-x=18$
The fourth contains all except u, that is, $s-u=21.$

Adding all these equations together, we have

$$4s-\ z-y-x-u\ =69$$

Or $4s-(z+y+x+u)=69$ (Art. 83.)

But $s=(z+y+x+u)$ by substitution.

Therefore $4s-s=69$; that is, $3s=69$, and $s=23$.

Then putting 23 for s, in the four equations in which it is first introduced, we have

$$\left.\begin{array}{l}23-z=13\\23-y=17\\23-x=18\\23-u=21\end{array}\right\}\text{Therefore}\left\{\begin{array}{l}z=23-13=10\\y=23-17=6\\x=23-18=5\\u=23-21=2.\end{array}\right.$$

Contrivances of this sort for facilitating the solution of particular problems, must be left to be furnished for the occasion, by the ingenuity of the learner. They are of a nature not to be taught by a system of rules.

216. In the resolution of equations containing several unknown quantities, there will often be an advantage in adopting the following method of notation.

The co-efficients of one of the unknown quantities are represented,

In the *first* equation, by a single letter, as a,

In the *second*, by the same letter marked with an accent, as a',

In the *third*, by the same letter with a *double* accent, as a'', &c.

The accented letters are called *a prime, a second, a third*, &c.

The co-efficients of the other unknown quantities, are represented by other letters marked in a similar manner; as are also the terms which consist of *known* quantities only.

Two equations containing the two unknown quantities x and y may be written thus,

$$ax+by=c$$
$$a'x+b'y=c'$$

Three equations containing x, y and z, thus,

$$ax+by+cz=d$$
$$a'x+b'y+c'z=d'$$
$$a''x+b''y+c''z=d''$$

Four equations containing x, y, z and u, thus,

$$ax+by+cz+du=e$$
$$a'x+b'y+c'z+d'u=e'$$
$$a''x+b''y+c''z+d''u=e''$$
$$a'''x+b'''y+c'''z+d'''u=e'''.$$

The same *letter* is made the co-efficient of the same unknown quantity, in different equations, that the co-efficients of the several unknown quantities may be distinguished, in any part of the calculation. But the letter is marked with different *accents*, because it actually stands for different quantities.

Thus we may put $a=4$, $a'=6$, $a''=10$, $a'''=20$, &c.

To find the value of x and y

1. In the equation, $ax+by=c$
2. And $a'x+b'y=c'$
3. Multiplying the 1st by b', (Art. 205.) $ab'x+bb'y=cb'$
4. Multiplying the 2d by b, $ba'x+bb'y=bc'$
5. Subtracting the 4th from the 3d, $ab'x-ba'x=cb'-bc'$
6. Dividing by $ab'-ba'$, (Art. 124.) $x=\dfrac{cb'-bc'}{ab'-ba'}$

By a similar process, $y=\dfrac{ac'-ca'}{ab'-ba'}$

The symmetry of these expressions is well calculated to fix them in the memory. The denominators are the same in both; and the numerators are like the denominators, except a change of one of the letters in each term.

But the particular advantage of this method is, that the expressions here obtained may be considered as *general solutions*, which give the values of the unknown quantities, in other equations, of a similar nature.

Thus if $10x+6y=100$
And $40x+4y=200$

Then putting $a=10$ $b=6$ $c=100$
$a'=40$ $b'=4$ $c'=200$

We have $x=\dfrac{cb'-bc'}{ab'-ba'}=\dfrac{100\times4-6\times200}{10\times4-6\times40}=4.$

And $y=\dfrac{ac'-ca'}{ab'-ba'}=\dfrac{10\times200-100\times40}{10\times4-6\times40}=10.$

Demonstration of Theorems.

217. Equations have been applied, in this and the preceding sections, to the solution of *problems*. They may be employed with equal advantage, in the demonstration of *theorems*. The principal difference, in the two cases, is in the order in which the steps are arranged.

In solving a problem, the object is to find the value of the unknown quantity, by disengaging it from all other quantities. But, in conducting a demonstration, it is necessary to bring the equation to that particular form which will express, in algebraic terms, the proposition to be proved.

Ex. 1. *Theorem.* Four times the product of any two numbers, is equal to the square of their sum, diminished by the square of their difference.

Let $x=$ the greater number, $s=$ their sum,
$y=$ the less, $d=$ their difference,
$p=$ their product.

Demonstration.

1.	By the notation	$x+y=s$
2.	By the notation	$x-y=d$
3.	By the notation	$xy=p$
4.	Adding the 1st and 2d	$2x=s+d$
5.	Subtracting the 2d from the 1st	$2y=s-d$
6.	Multiplying the 4th and 5th	$4xy=(s+d)(s-d)$
7.	By substitution and Art. 111	$4p=s^2-d^2$.

The last equation expressed in words is the proposition which was to be demonstrated.

It will be seen that the demonstration consists in first expressing p, s, and d in terms of x and y by means of three equations, and then eliminating x and y, so as to obtain an equation in terms of p, s and d only. The final equation must be of this kind, since the proposition relates not to the quantities x and y, but only to their product, sum and difference.

The theorem is applicable to any two numbers whatever. For the particular values of x and y will make no difference in the nature of the proof.

Thus $4\times 8\times 6=(8+6)^2-(8-6)^2=192$.

And $4\times 10\times 6=(10+6)^2-(10-6)^2=240$.

And $4\times 12\times 10=(12+10)^2-(12-10)^2=480$.

Theorem 2. The sum of the squares of two numbers is equal to the square of their sum, diminished by twice their product.

218. General propositions are also *discovered*, in an expeditious manner, by means of equations. The relations of quantities may be presented to our view, in a great variety of ways, by the several changes through which a given equation may be made to pass. Each step in the process will contain a distinct proposition.

Let s and d be the sum and difference of two quantities x and y.

1. Then $s=x+y$
2. And $d=x-y$
3. Dividing the first by 2, $\frac{1}{2}s=\frac{1}{2}x+\frac{1}{2}y$
4. Dividing the 2d by 2, $\frac{1}{2}d=\frac{1}{2}x-\frac{1}{2}y$
5. Adding the 3d and 4th, $\frac{1}{2}s+\frac{1}{2}d=\frac{1}{2}x+\frac{1}{2}x=x$
6. Subtracting the 4th from the 3d, $\frac{1}{2}s-\frac{1}{2}d=\frac{1}{2}y+\frac{1}{2}y=y$.

That is,

Half the sum of two quantities increased by half their difference is equal to the greater; and

Half their sum diminished by half their difference is equal to the less.

SECTION IX.

INVOLUTION AND EVOLUTION.

ART. **219.** WHEN *a quantity is multiplied into* ITSELF, *the* PRODUCT *is called a* POWER.

For the notation of powers see Arts. 36, 7.

The scheme of notation by exponents has the peculiar advantage of enabling us to express an *unknown* power. For this purpose the index is a *letter*, instead of a numerical figure. In the solution of a problem, a quantity may occur, which we know to be *some* power of another quantity. But it may not be yet ascertained whether it is a square, a cube, or some higher power. Thus in the expression a^x, the index x denotes that a is involved to some power, though it does not determine *what* power. So b^m and d^n are powers of b and d; and are read the mth power of b, and the nth power of d. When the value of the index is found, a *number* is generally substituted for the letter. Thus if $m=3$ then $b^m=b^3$; but if $m=5$, then $b^m=b^5$.

220. The method of expressing powers by exponents is also of great advantage in the case of *compound* quantities. Thus $\overline{a+b+d}|^3$ or $\overline{a+b+d}^3$ or $(a+b+d)^3$, is $(a+b+d)\times(a+b+d)\times(a+b+d)$ that is, the cube of $(a+b+d)$. But this involved at length would be

$$a^3+3a^2b+3a^2d+3ab^2+6abd+3ad^2+b^3+3b^2d+3bd^2+d^3.$$

221. If we take a series* of powers whose indices increase or decrease by 1, we shall find that the powers themselves increase by a *common multiplier*, or decrease by a *common divisor;* and that this multiplier or divisor is the original quantity from which the powers are raised.

Thus in the series	$aaaaa$,	$aaaa$,	aaa,	aa,	1;
Or	a^5	a^4	a^3	a^2	a^1;

* NOTE.—The term *series* is applied to a number of quantities succeeding each other, in some regular order. It is not confined to any particular law of increase or decrease.

the indices counted from right to left are 1, 2, 3, 4, 5; and the common difference between them is a unit. If we begin on the *right* and *multiply* by a, we produce the several powers, in succession, from right to left.

Thus $a \times a = a^2$ the second term. And $a^3 \times a = a^4$
$a^2 \times a = a^3$ the third term. $a^4 \times a = a^5$, &c.

If we begin on the *left*, and *divide* by a,

We have $a^5 \div a = a^4$ And $a^3 \div a = a^2$
$a^4 \div a = a^3$ $a^2 \div a = a^1$

222. But this division may be carried still farther; and we shall then obtain a new set of quantities.

Thus $a \div a = \frac{a}{a} = 1$. (Art. 123.) $\frac{1}{a} \div a = \frac{1}{aa}$. (Art. 166.)
$1 \div a = \frac{1}{a}$ $\frac{1}{aa} \div a = \frac{1}{aaa}$, &c.

The whole series then

is $aaaaa$, $aaaa$, aaa, aa, a, 1, $\frac{1}{a}$, $\frac{1}{aa}$, $\frac{1}{aaa}$, &c.

Or a^5, a^4, a^3, a^2, a, 1, $\frac{1}{a}$, $\frac{1}{a^2}$, $\frac{1}{a^3}$, &c.

Here the quantities on the *right* of 1, are the *reciprocals* of those on the *left*. (Art. 43.) The former, therefore, may be properly called *reciprocal powers* of a; while the latter may be termed, for distinction's sake, *direct powers* of a. It may be added, that the powers on the left are also the reciprocals of those on the right.

For $1 \div \frac{1}{a} = 1 \times \frac{a}{1} = a$. (Art. 165.) And $1 \div \frac{1}{a^3} = a^3$.
$1 \div \frac{1}{a^2} = 1 \times \frac{a^2}{1} = a^2$. $1 \div \frac{1}{a^4} = a^4$, &c.

223. The same plan of notation is applicable to *compound* quantities. Thus from $a+b$, we have the series,

$(a+b)^3$, $(a+b)^2$, $(a+b)$, 1, $\frac{1}{(a+b)}$, $\frac{1}{(a+b)^2}$, $\frac{1}{(a+b)^3}$, &c.

224. For the convenience of calculation, another form of notation is given to reciprocal powers.

According to this, $\frac{1}{a}$ or $\frac{1}{a^1}=a^{-1}$. And $\frac{1}{aaa}$ or $\frac{1}{a^3}=a^{-3}$.

$\frac{1}{aa}$ or $\frac{1}{a^2}=a^{-2}$. $\frac{1}{aaaa}$ or $\frac{1}{a^4}=a^{-4}$, &c.

And to make the indices a complete series, with 1 for the common difference, the term $\frac{a}{a}$ or 1, which is considered as *no* power, is written a^0. Though a^0 has no more effect as a factor than unity, yet it is sometimes expedient to retain it, in connection with other letters; for the purpose of indicating, that it is the result of a division, where equal powers of a quantity had been in both the dividend and divisor.

Thus $\frac{a^m}{a^m}=a^{m-m}=a^0$. And $\frac{6b^4c^3d^5}{2b^4c^2d^5}=3b^0cd^0$.

The powers both direct and reciprocal* then,

Instead of $aaaa,\ aaa,\ aa,\ a,\ \frac{a}{a},\ \frac{1}{a},\ \frac{1}{aa},\ \frac{1}{aaa},\ \frac{1}{aaaa}$, &c.

Will be $a^4,\ a^3,\ a^2,\ a^1,\ a^0,\ a^{-1},\ a^{-2},\ a^{-3},\ a^{-4}$, &c.

Or $a^{+4},\ a^{+3},\ a^{+2},\ a^{+1},\ a^0,\ a^{-1},\ a^{-2},\ a^{-3},\ a^{-4}$, &c.

And the indices taken by themselves will be,

$+4, +3, +2, +1, +0, -1, -2, -3, -4$, &c.

225. The root of a power may be expressed by more letters than one.

Thus $aa\times aa$, or $\overline{aa}|^2$ is the second power of aa.

And $aa\times aa\times aa$, or $\overline{aa}|^3$ is the third power of aa, &c.

Hence a certain power of one quantity, may be a different power of another quantity. Thus a^4 is the second power of a^2, and the fourth power of a.

226. All the powers of 1 are the same. For 1×1, or $1\times1\times1$, &c. is still 1.

---

* See Note E.

227. Involution is finding any power of a quantity, by multiplying it into itself. The reason of the following general rule is manifest, from the nature of powers.

Multiply the quantity into itself, till it is taken as a factor, as many times as there are units in the index of the power to which the quantity is to be raised.

This rule comprehends all the instances which can occur in involution. But it will be proper to give an explanation of the manner in which it is applied to particular cases.

228. A single letter is involved, by giving it the index of the proposed power; or by repeating it as many times, as there are units in that index.

The 4th power of a, is a^4 or $aaaa$.

The 6th power of y, is y^6, or $yyyyyy$.

The nth power of x, is x^n or $xxx \ldots n$ times repeated.

229. The method of involving a quantity which consists of several *factors*, depends on the principle, that *the power of the product of several factors is equal to the product of their powers.*

Thus $(ay)^2=a^2y^2$. For by Art. 227; $(ay)^2=ay\times ay$.

But $ay\times ay=ayay=aayy=a^2y^2$.

So $(bmx)^3=bmx\times bmx\times bmx=bbbmmmxxx=b^3m^3x^3$.

And $(ady)^n=ady\times ady\times ady \ldots n$ times$=a^nd^ny^n$.

In finding the power of a product, therefore, we may either involve the whole at once; or we may involve each of the factors separately, and then multiply their several powers into each other.

Ex. 1. The 4th power of dhy, is $(dhy)^4$, or $d^4h^4y^4$.

2. The 3d power of $4b$, is $(4b)^3$, or 4^3b^3, or $64b^3$.

3. The nth power of $6ad$, is $(6ad)^n$, or $6^na^nd^n$.

4. The 3d power of $3m\times 2y$, is $(3m\times 2y)^3$, or $27m^3\times 8y^3$.

230. A compound quantity consisting of terms connected by + and −, is involved by an actual multiplication of its several parts. Thus,

$$
\begin{array}{l}
(a+b)^1 = a + b, \text{ the first power.} \\
\qquad\quad a + b \\
\hline
\qquad\quad a^2 + ab \\
\qquad\qquad + ab + b^2 \\
\hline
(a+b)^2 = a^2 + 2ab + b^2, \text{ the second power of } (a+b). \\
\qquad\quad a + b \\
\hline
\qquad\quad a^3 + 2a^2b + ab^2 \\
\qquad\qquad + a^2b + 2ab^2 + b^3 \\
\hline
(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3, \text{ the third power, \&c.}
\end{array}
$$

2. The square of $a-b$, is $a^2-2ab+b^2$.
3. The cube of $a+1$ is a^3+3a^2+3a+1.
4. The square of $a+b+h$, is $a^2+2ab+2ah+b^2+2bh+h^2$.
5. Required the cube of $a+2d+3$.
6. Required the 4th power of $b+2$.
7. Required the 5th power of $x+1$.
8. Required the 6th power of $1-b$.

For the method of finding the higher powers of binomials, see one of the succeeding sections.

231. For many purposes, it will be sufficient to express the powers of compound quantities by *exponents*, without an actual multiplication.

Thus the square of $a+b$, is $\overline{a+b}|^2$ or $(a+b)^2$. (Art. 220.)
The nth power of $bc+8+x$, is $(bc+8+x)^n$.

In cases of this kind, the vinculum must be drawn over *all* the terms of which the compound quantity consists.

232. But if the root consists of several *factors*, the vinculum which is used in expressing the power, may either extend over the whole; or may be applied to each of the factors separately, as convenience may require.

Thus the square of $\overline{a+b}\times\overline{c+d}$, is either

$$\overline{(a+b)\times(c+d)}|^2 \text{ or } \overline{a+b}|^2\times\overline{c+d}|^2.$$

For the first of these expressions is the square of the product of the two factors, and the last is the product of their squares. But one of these is equal to the other. (Art. 229.)

The cube of $a\times\overline{b+d}$ is $(a\times\overline{b+d})^3$ or $a^3\times(b+d)^3$.

When a quantity whose power has been expresssed by a vinculum and an index, is afterwards involved by an actual multiplication of the terms, it is said to be *expanded.*

Thus $(a+b)^2$, when expanded, becomes $a^2+2ab+b^2$.
And $(a+b+h)^2$, becomes $a^2+2ab+2ah+b^2+2bh+h^2$.

233. With respect to the *sign* which is to be prefixed to quantities involved, it is important to observe, that *when the root is positive, all its powers are positive also; but when the root is negative, the* ODD *powers are negative, while the* EVEN *powers are positive.*

For the proof of this, see Art. 103.

The 2d power of $-a$ is $+a^2$
The 3d power is $-a^3$
The 4th power is $+a^4$
The 5th power is $-a^5$, &c.

Hence any *odd* power has the same sign as its root. But an *even* power is positive, whether its root is positive or negative.

Thus $+a\times+a=a^2$
And $-a\times-a=a^2$.

234. *A quantity which is already a power, is involved by multiplying its index, into the index of the power to which it is to be raised.*

1. The 3d power of a^2, is $a^{2\times3}=a^6$.

For $a^2=aa$: and the cube of aa is $aa\times aa\times aa=aaaaaa=a^6$; which is the 6th power of a, but the 3d power of a^2.

2. The 4th power of a^3b^2, is $a^{3\times4}b^{2\times4}=a^{12}b^8$.
3. The 3d power of $4\,a^2x$, is $64\,a^6x^3$.
4. The 4th power of $2a^3\times3x^2d$, is $16a^{12}\times81x^8d^4$.
5. The 5th power of $(a+b)^2$, is $(a+b)^{10}$.

6. The nth power of a^3, is a^{3n}.
7. The nth power of $(x-y)^m$, is $(x-y)^{mn}$.
8. $\overline{a^3+b^3}|^2=a^6+2a^3b^3+b^6$. (Art. 109.)
9. $\overline{a^3\times b^3}|^2=a^6\times b^6$. 10. $(a^3b^2h^4)^3=a^9b^6h^{12}$.

235. The rule is equally applicable to powers whose exponents are *negative.*

Ex. 1. The 3d power of a^{-2}, is $a^{-2\times3}=a^{-6}$.

For $a^{-2}=\frac{1}{aa}$, (Art. 224.) And the 3d power of this is

$$\frac{1}{aa}\times\frac{1}{aa}\times\frac{1}{aa}=\frac{1}{aaaaaa}=\frac{1}{a^6}=a^{-6}.$$

2. The 4th power of a^2b^{-3} is a^8b^{-12}, or $\frac{a^8}{b^{12}}$.
3. The cube of $2x^ny^{-m}$, is $8x^{3n}y^{-3m}$.
4. The square of b^3x^{-1}, is b^6x^{-2}.
5. The nth power of x^{-m}, is x^{-mn}, or $\frac{1}{x^{mn}}$.

236. It must be observed here, as in Art. 233, that if the sign which is *prefixed* to the power be $-$, it must be changed to $+$, whenever the index becomes an even number.

Ex. 1. The square of $-a^3$, is $+a^6$. For the square of $-a^3$, is $-a^3\times-a^3$, which, according to the rules for the signs in multiplication, is $+a^6$.

2. But the cube of $-a^3$ is $-a^9$. For $-a^3\times-a^3\times-a^3=-a^9$.
3. The square of $-x^n$, is $+x^{2n}$.
4. The nth power of $-a^3$, is $\pm a^{3n}$.

Here the power will be positive or negative, according as the number which n represents is even or odd.

237. *A* FRACTION *is involved by involving both the numerator and the denominator.*

1. The square of $\frac{a}{b}$ is $\frac{a^2}{b^2}$. For, by the rule for the multiplication of fractions, (Art. 158,)

$$\frac{a}{b}\times\frac{a}{b}=\frac{aa}{bb}=\frac{a^2}{b^2}.$$

2. The 2d, 3d, and nth powers of $\frac{1}{a}$, are $\frac{1}{a^2}$, $\frac{1}{a^3}$ and $\frac{1}{a^n}$.

3. The cube of $\frac{2xr^2}{3y}$, is $\frac{8x^3r^6}{27y^3}$.

4. The nth power of $\frac{x^2r}{ay^m}$, is $\frac{x^{2n}r^n}{a^ny^{mn}}$.

5. The square of $\frac{-a^3\times(d+m)}{(x+1)^3}$, is $\frac{a^6\times(d+m)^2}{(x+1)^6}$.

6. The cube of $\frac{-a^{-1}}{x^{-3}}$, is $\frac{-a^{-3}}{x^{-9}}$. (Art. 235.)

238. Examples of *binomials*, in which one of the terms is a fraction.

1. Find the square of $x+\frac{1}{2}$, and $x-\frac{1}{2}$, as in Art. 109.

$$\begin{array}{ll} x+\frac{1}{2} & \qquad x-\frac{1}{2} \\ x+\frac{1}{2} & \qquad x-\frac{1}{2} \\ \hline x^2+\frac{1}{2}x & \qquad x^2-\frac{1}{2}x \\ \quad +\frac{1}{2}x+\frac{1}{4} & \qquad \quad -\frac{1}{2}x+\frac{1}{4} \\ \hline x^2+x+\frac{1}{4} & \qquad x^2-x+\frac{1}{4} \end{array}$$

2. The square of $a+\frac{2}{3}$, is $a^2+\frac{4a}{3}+\frac{4}{9}$.

3. The square of $x+\frac{b}{2}$, is $x^2+bx+\frac{b^2}{4}$.

4. The square of $x-\frac{b}{m}$, is $x^2-\frac{2bx}{m}+\frac{b^2}{m^2}$.

239. It has been shown, (Art. 168,) that a *fractional co-efficient* may be transferred from the numerator to the denominator of a fraction, or from the denominator to the numerator. By recurring to the scheme of notation for reciprocal powers, (Art. 224,) it will be seen that *any factor* may also be transferred, *if the sign of its index be changed.*

1. Thus, in the fraction $\frac{ax^{-2}}{y}$, we may transfer x from the numerator to the denominator.

For $\frac{ax^{-2}}{y}=\frac{a}{y}\times x^{-2}=\frac{a}{y}\times\frac{1}{x^2}=\frac{a}{yx^2}$.

2. In the fraction $\frac{a}{by^3}$, we may transfer y from the denominator to the numerator.

For $\frac{a}{by^3}=\frac{a}{b}\times\frac{1}{y^3}=\frac{a}{b}\times y^{-3}=\frac{ay^{-3}}{b}$.

3. $\frac{da^{-4}}{x^3}=\frac{d}{x^3a^4}$. 4. $\frac{b}{ay^n}=\frac{by^{-n}}{a}$.

240. In the same manner, we may transfer a factor which has a positive index in the numerator, or a negative index in the denominator.

1. Thus $\frac{ax^3}{b}=\frac{a}{bx^{-3}}$. For x^3 is the reciprocal of x^{-3}, (Arts. 222, 224,) that is, $x^3=\frac{1}{x^{-3}}$. Therefore, $\frac{ax^3}{b}=\frac{a}{bx^{-3}}$.

2. $\frac{h}{by^{-2}}=\frac{hy^2}{b}$. 3. $\frac{ad^2}{xy^{-2}}=\frac{ay^2}{xd^{-2}}$.

241. Hence the denominator of any fraction may be entirely removed, or the numerator may be reduced to a unit, without altering the value of the expression.

1. Thus $\frac{a}{b}=\frac{1}{ba^{-1}}$, or ab^{-1}.

2. $\frac{x^{-2}}{b^{-n}}=\frac{1}{x^2b^{-n}}$, or b^nx^{-2}.

3. $\frac{x^4a^{-m}}{b^nc^{-3}}=\frac{1}{b^na^mx^{-4}c^{-3}}$, or $c^3x^4a^{-m}b^{-n}$.

Addition and Subtraction of Powers.

242. It is obvious that powers may be added, like other quantities, *by writing them one after another with their signs.*

Thus the sum of a^3 and b^2, is a^3+b^2.

And the sum of a^2-b^n and h^5-d^4, is $a^2-b^n+h^5-d^4$

243. The *same powers of the same letters* are *like quantities;* (Art. 39,) and their co-efficients may be added or subtracted, as in Arts. 67, 69.

Thus the sum of $2a^2$ and $3a^2$ is $5a^2$.

It is as evident that twice the square of a, and three times the square of a, are five times the square of a, as that twice a and three times a, are five times a.

To	$-3x^6y^5$	$3b^m$	$3a^4y^n$	$-5a^3h^6$	$3(a+y)^n$
Add	$-2x^6y^5$	$6b^m$	$-7a^4y^n$	$6a^3h^6$	$4(a+y)^n$
Sum	$-5x^6y^5$		$-4a^4y^n$		$7(a+y)^n$

244. But powers of *different letters* and *different powers* of the *same letter*, must be added by writing them down with their signs.

The sum of a^2 and a^3 is a^2+a^3.

It is evident that the square of a, and the cube of a, are neither twice the square of a, nor twice the cube of a.

The sum of a^3b^n and $3a^5b^6$, is $a^3b^n+3a^5b^6$.

245. *Subtraction* of powers is to be performed in the same manner as addition, except that the signs of the subtrahend are to be changed according to Art. 75.

From	$2a^4$	$-3b^n$	$3h^2b^6$	a^3b^n	$5(a-h)^6$
Subtract	$-6a^4$	$4b^n$	$4h^2b^6$	a^3b^n	$2(a-h)^6$
Difference	$8a^4$		$-h^2b^6$		$3(a-h)^6$

MULTIPLICATION OF POWERS.

246. Powers may be multiplied, like other quantities, by writing the factors one after another, either with, or without, the sign of multiplication between them. (Art. 89.)

Thus the product of a^3 into b^2, is a^3b^2, or $aaabb$.

Multiply	x^{-3}	h^2b^{-n}	$3a^6y^2$	dh^3x^{-n}	$a^2b^3y^2$
Into	a^m	a^4	$-2x$	$4by^4$	a^3b^2y
Product	a^mx^{-3}		$-6a^6xy^2$		$a^2b^3y^2a^3b^2y$

The product in the last example, may be abridged, by bringing together the letters which are repeated.

It will then become $a^5b^5y^3$.

The reason of this will be evident, by recurring to the series of powers in Art. 224, viz.

$$a^{+4},\ a^{+3},\ a^{+2},\ a^{+1},\ a^0,\ a^{-1},\ a^{-2},\ a^{-3},\ a^{-4},\ \&c.$$

Or, which is the same,

$$aaaa,\ aaa,\ aa,\ a,\ 1,\ \frac{1}{a},\ \frac{1}{aa},\ \frac{1}{aaa},\ \frac{1}{aaaa},\ \&c.$$

By comparing the several terms with each other, it will be seen that if any two or more of them be multiplied together, their product will be a power whose exponent is the *sum* of the exponents of the factors.

Thus $a^2\times a^3=aa\times aaa=aaaaa=a^5$. So $a^n\times a^m=a^{n+m}$. Hence,

247. *Powers of the same root may be multiplied, by adding their exponents.*

Thus $a^2\times a^6=a^{2+6}=a^8$. And $x^3\times x^2\times x=x^{3+2+1}=x^6$.

Mult. $x^3+x^2y+xy^2+y^3$ into $x-y$. Ans. x^4-y^4.

Mult. $4x^2y+3xy-1$ into $2x^2-x$.

Mult. x^3+x-5 into $2x^2+x+1$.

The rule is equally applicable to powers whose exponents are *negative*.

1. Thus $a^{-2}\times a^{-3}=a^{-5}$. That is $\dfrac{1}{aa}\times\dfrac{1}{aaa}=\dfrac{1}{aaaaa}$.

2. $y^{-n}\times y^{-m}=y^{-n-m}$. That is $\dfrac{1}{y^n}\times\dfrac{1}{y^m}=\dfrac{1}{y^ny^m}$.

3. $-a^{-2}\times a^{-3}=-a^{-5}$. 4. $a^{-2}\times a^3=a^{3-2}=a^1$.

5. $a^{-n}\times a^m=a^{m-n}$. 6. $y^{-2}\times y^2=y^0=1$.

Division of Powers.

248. Powers may be divided, like other quantities, by rejecting from the dividend a factor equal to the divisor; or by placing the divisor under the dividend, in the form of a fraction.

Thus the quotient of a^3b^2 divided by b^2, is a^3.

Divide	$9a^3y^4$	$12b^3x^n$	$a^2b+3a^2y^4$	$d\times(a-h+y)^3$
By	$-3a^3$	$2b^3$	a^2	$(a-h+y)^3$
Quotient	$-3y^4$		$b+3y^4$	d

The quotient of a^5 divided by a^3, is $\frac{a^5}{a^3}$. But this is equal to a^2. For, in the series

$$a^{+4},\ a^{+3},\ a^{+2},\ a^{+1},\ a^0,\ a^{-1},\ a^{-2},\ a^{-3},\ a^{-4},\ \&c.$$

if any term be divided by another, the index of the quotient will be equal to the *difference* between the index of the dividend and that of the divisor.

Thus $a^5\div a^3=\frac{aaaaa}{aaa}=a^2$. And $a^m\div a^n=\frac{a^m}{a^n}=a^{m-n}$.

Hence,

249. ***A power may be divided by another power of the same root, by subtracting the index of the divisor from that of the dividend.***

Thus $y^3\div y^2=y^{3-2}=y^1$. That is $\frac{yyy}{yy}=y$.

And $a^{n+1}\div a=a^{n+1-1}=a^n$. That is $\frac{aa^n}{a}=a^n$.

And $x^n\div x^n=x^{n-n}=x^0=1$. That is $\frac{x^n}{x^n}=1$.

Divide	y^{2m}	b^6	$8a^{n+m}$	a^{n+3}	$12(b+y)^n$
By	y^m	b^3	$4a^m$	a^3	$3(b+y)^3$
Quotient	y^m		$2a^n$		$4(b+y)^{n-3}$

250. The rule is equally applicable to powers whose exponents are *negative*.

1. The quotient of a^{-5} by a^{-3}, is a^{-2}.

That is $\frac{1}{aaaaa}\div\frac{1}{aaa}=\frac{1}{aaaaa}\times\frac{aaa}{1}=\frac{aaa}{aaaaa}=\frac{1}{aa}$.

2. $-x^{-5}\div x^{-3}=-x^{-2}$. That is $\frac{1}{-x^5}\div\frac{1}{x^3}=\frac{x^3}{-x^5}=\frac{1}{-x^2}$.

3. $h^2 \div h^{-1} = h^{2+1} = h^3$. That is $h^2 \div \frac{1}{h} = h^2 \times \frac{h}{1} = h^3$.

4. $6a^n \div 2a^{-3} = 3a^{n+3}$.

5. $ba^3 \div a = ba^2$.

6. $b^3 \div b^5 = b^{3-5} = b^{-2}$.

7. $a^4 \div a^7 = a^{-3}$.

8. $(a^3+y^3)^m \div (a^3+y^3)^n = (a^3+y^3)^{m-n}$.

9. $(b+x)^n \div (b+x) = (b+x)^{n-1}$.

The multiplication and division of powers, by adding and subtracting their indices, should be made very familiar; as they have numerous and important applications, in the higher branches of algebra.

Evolution.

251. If a quantity is multiplied into itself, the product is a *power*. On the contrary, if a quantity is resolved into any number of *equal factors*, each of these is a *root* of that quantity.

Thus b is the root of bbb; because bbb may be resolved into the three equal factors, b, and b, and b.

In subtraction, a quantity is resolved into *two parts*.

In division, a quantity is resolved into *two factors*.

In evolution, a quantity is resolved into *equal factors*.

252. *A root of a quantity, then, is a factor, which multiplied into itself a certain number of times, will produce that quantity.*

The number of times the root must be taken as a factor, to produce the given quantity, is denoted by the name of the root.

Thus 2 is the 4th root of 16; because $2\times2\times2\times2=16$, where two is taken *four* times as a factor, to produce 16.

So a^3 is the square root of a^6; for $a^3\times a^3=a^6$.

And a^2 is the cube root of a^6; for $a^2\times a^2\times a^2=a^6$.

And a is the 6th root of a^6; for $a\times a\times a\times a\times a\times a=a^6$.

Powers and roots are correlative terms. If one quantity is a power of another, the latter is a root of the former. As b^3 is the cube of b, b is the cube root of b^3.

253. There are two methods in use, for expressing the roots of quantities; one by means of the radical sign $\sqrt{}$, and the other by a fractional index. The latter is generally to be preferred; but the former has its uses on particular occasions.

When a root is expressed by the radical sign, the sign is placed over the given quantity, in this manner, $\sqrt{a}$.

Thus $\sqrt[2]{a}$ is the 2d or square root of a.

$\sqrt[3]{a}$ is the 3d or cube root.

$\sqrt[n]{a}$ is the nth root.

And $\sqrt[n]{a+y}$ is the nth root of $a+y$.

254. The figure placed over the radical sign, denotes the number of factors into which the given quantity is resolved; in other words, the number of times the root must be taken as a factor to produce the given quantity.

So that $\sqrt[2]{a}\times\sqrt[2]{a}=a$.

And $\sqrt[3]{a}\times\sqrt[3]{a}\times\sqrt[3]{a}=a$.

And $\sqrt[n]{a}\times\sqrt[n]{a}\ldots.n$ times $=a$.

The figure for the *square* root is commonly omitted; $\sqrt{a}$ being put for $\sqrt[2]{a}$. Whenever, therefore, the radical sign is used without a figure, the square root is to be understood.

255. When a figure or letter is *prefixed* to the radical sign, without any character between them, the two quantities are to be considered as *multiplied* together.

Thus $2\sqrt{a}$, is $2\times\sqrt{a}$, that is, 2 multiplied into the root of a, or, which is the same thing, *twice* the root of a.

And $x\sqrt{b}$, is $x\times\sqrt{b}$, or x times the root of b.

When no co-efficient is prefixed to the radical sign, 1 is always to be understood; $\sqrt{a}$ being the same as $1\sqrt{a}$, that is, *once* the root of a.

256. The method of expressing roots by radical signs, has no very apparent connection with the other parts of the scheme of algebraic notation. But the plan of indicating them by *fractional indices*, is derived directly from the mode of expressing *powers* by *integral* indices. To explain this, let a^6 be a given quantity. If the index be divided into any number of equal parts, each of these will be the index of a root of a^6.

Thus the *square* root of a^6 is a^3. For, according to the definition, (Art. 252,) the square root of a^6 is a factor, which multiplied into itself will produce a^6. But $a^3 \times a^3 = a^6$. (Art. 247.) Therefore, a^3 is the square root of a^6. The index of the given quantity a^6, is here divided into the two equal parts, 3 and 3. Of course, the quantity itself is resolved into the two equal factors, a^3 and a^3.

The *cube* root of a^6 is a^2. For $a^2 \times a^2 \times a^2 = a^6$.

Here the index is divided into *three* equal parts, and the quantity itself resolved into three equal factors.

The square root of a^2 is a^1 or a. For $a \times a = a^2$.

By extending the same plan of notation, *fractional indices* are obtained.

Thus, in taking the square root of a^1 or a, the index 1 is divided into two equal parts, $\frac{1}{2}$ and $\frac{1}{2}$; and the root is $a^{\frac{1}{2}}$.

On the same principle,

The cube root of a, is $a^{\frac{1}{3}} = \sqrt[3]{a}$.

The nth root, is $a^{\frac{1}{n}} = \sqrt[n]{a}$, &c.

And the nth root of $a+x$, is $(a+x)^{\frac{1}{n}} = \sqrt[n]{a+x}$.

257. In all these cases, the denominator of the fractional index, expresses the number of factors into which the given quantity is resolved.

So that $a^{\frac{1}{3}} \times a^{\frac{1}{3}} \times a^{\frac{1}{3}} = a$. And $a^{\frac{1}{n}} \times a^{\frac{1}{n}} \ldots . n$ times $= a$.

258. It follows from this plan of notation, that

$$a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}}. \text{ For } a^{\frac{1}{2}+\frac{1}{2}} = a^1 \text{ or } a.$$

$$a^{\frac{1}{3}} \times a^{\frac{1}{3}} \times a^{\frac{1}{3}} = a^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = a^1, \text{ \&c.}$$

where the multiplication is performed in the same manner as the multiplication of powers, that is, by *adding the indices*.

259. Every root as well as every power of 1 is 1. (Art. 226.) For a root is a factor, which multiplied into itself will produce the given quantity. But no factor except 1 can produce 1, by being multiplied into itself.

So that 1^n, 1, $\sqrt{1}$, $\sqrt[n]{1}$, &c. are all equal.

260. *Negative* indices are used in the notation of roots, as well as of powers. See Art. 224.

Thus $\frac{1}{a^{\frac{1}{2}}}=a^{-\frac{1}{2}}$ $\frac{1}{a^{\frac{1}{3}}}=a^{-\frac{1}{3}}$ $\frac{1}{a^{\frac{1}{n}}}=a^{-\frac{1}{n}}$.

Powers of Roots.

261. It has been shown in what manner any power or root may be expressed by means of an index. The index of a power is a whole number. That of a root is a fraction whose numerator is 1. There is also another class of quantities which may be considered, either as powers of roots, or roots of powers.

Suppose $a^{\frac{1}{2}}$ is multiplied into itself, so as to be repeated three times as a factor.

The product $a^{\frac{1}{2}+\frac{1}{2}+\frac{1}{2}}$ or $a^{\frac{3}{2}}$ (Art. 258,) is evidently the cube of $a^{\frac{1}{2}}$, that is, the cube of the square root of a. This fractional index denotes, therefore, *a power of a root.* The denominator expresses the root, and the numerator the power. The denominator shows into how many equal factors or roots the given quantity is resolved; and the numerator shows how many of these roots are to be multiplied together.

Thus $a^{\frac{4}{3}}$ is the 4th power of the cube root of a.

The denominator shows that a is resolved into the three factors or roots $a^{\frac{1}{3}}$, and $a^{\frac{1}{3}}$, and $a^{\frac{1}{3}}$. And the numerator shows that four of these are to be multiplied together; which will produce the fourth power of $a^{\frac{1}{3}}$; that is,

$$a^{\frac{1}{3}}\times a^{\frac{1}{3}}\times a^{\frac{1}{3}}\times a^{\frac{1}{3}}=a^{\frac{4}{3}}.$$

262. As $a^{\frac{3}{2}}$ is a power of a root, so it is *a root of a power.* Let a be raised to the third power a^3. The square root of this is $a^{\frac{3}{2}}$. For the root of a^3 is a quantity which multiplied into itself will produce a^3.

But according to Art. 261, $a^{\frac{3}{2}}=a^{\frac{1}{2}}\times a^{\frac{1}{2}}\times a^{\frac{1}{2}}$; and this multiplied into itself, is

$$a^{\frac{1}{2}}\times a^{\frac{1}{2}}\times a^{\frac{1}{2}}\times a^{\frac{1}{2}}\times a^{\frac{1}{2}}\times a^{\frac{1}{2}}=a^3.$$

Therefore $a^{\frac{3}{2}}$ is the square root of the cube of a.

In the same manner, it may be shown that $a^{\frac{m}{n}}$ is the mth power of the nth root of a; or the nth root of the mth power: that is, *a root of a power is equal to the same power of the same root.* For instance, the fourth power of the cube root of a, is the same as the cube root of the fourth power of a.

263. Roots, as well as powers, of the same letter, may be multiplied by *adding their exponents.* It will be easy to see, that the same principle may be extended to powers of roots, when the exponents have a common denominator.

Thus $a^{\frac{2}{7}}\times a^{\frac{3}{7}}=a^{\frac{2}{7}+\frac{3}{7}}=a^{\frac{5}{7}}$.

For $\quad a^{\frac{2}{7}}=a^{\frac{1}{7}}\times a^{\frac{1}{7}}$.

And $\quad a^{\frac{3}{7}}=a^{\frac{1}{7}}\times a^{\frac{1}{7}}\times a^{\frac{1}{7}}$.

Therefore $a^{\frac{2}{7}}\times a^{\frac{3}{7}}=a^{\frac{1}{7}}\times a^{\frac{1}{7}}\times a^{\frac{1}{7}}\times a^{\frac{1}{7}}\times a^{\frac{1}{7}}=a^{\frac{5}{7}}$.

264. The value of a quantity is not altered, by applying to it a fractional index whose numerator and denominator are equal.

Thus $a=a^{\frac{2}{2}}=a^{\frac{3}{3}}=a^{\frac{n}{n}}$. For the denominator shows that a is resolved into a certain number of factors; and the numerator shows that all these factors are included in $a^{\frac{n}{n}}$.

Thus $a^{\frac{3}{3}}=a^{\frac{1}{3}}\times a^{\frac{1}{3}}\times a^{\frac{1}{3}}$, which is equal to a.

And $a^{\frac{n}{n}}=a^{\frac{1}{n}}\times a^{\frac{1}{n}}\times a^{\frac{1}{n}}\ldots\ldots n$ times.

On the other hand, when the numerator of a fractional index becomes equal to the denominator, the expression may be rendered more simple by *rejecting* the index.

Instead of $a^{\frac{n}{n}}$, we may write a.

265. The index of a power or root may be exchanged, for any other index of the same value.

Instead of $a^{\frac{2}{3}}$, we may put $a^{\frac{4}{6}}$.

For in the latter of these expressions, a is supposed to be resolved into *twice* as many factors as in the former; and the numerator shows that *twice* as many of these factors are

to be multiplied together. So that the whole value is not altered.

Thus $x^{\frac{2}{3}}=x^{\frac{4}{6}}=x^{\frac{6}{9}}$, &c. that is, the square of the cube root is the same, as the fourth power of the sixth root, the sixth power of the ninth root, &c.

So $a^2=a^{\frac{4}{2}}=a^{\frac{6}{3}}=a^{\frac{2n}{n}}$. For the value of each of these indices is 2.

266. From the preceding article, it will be easily seen, that a fractional index may be expressed in *decimals*.

1. Thus $a^{\frac{1}{2}}=a^{\frac{5}{10}}$, or $a^{0.5}$; that is, the square root is equal to the 5th power of the tenth root.

2. $a^{\frac{1}{4}}=a^{\frac{25}{100}}$, or $a^{0.25}$; that is, the fourth root is equal to the 25th power of the 100th root.

3. $a^{\frac{2}{5}}=a^{0.4}$

4. $a^{\frac{7}{2}}=a^{3.5}$

5. $a^{\frac{8}{5}}=a^{1.6}$

6. $a^{\frac{11}{4}}=a^{2.75}$.

In many cases, however, the decimal can be only an *approximation* to the true index.

Thus $a^{\frac{1}{3}}=a^{0.3}$ nearly. $a^{\frac{1}{3}}=a^{0.333333}$ very nearly.

In this manner, the approximation may be carried to any degree of exactness which is required.

Thus $a^{\frac{5}{3}}=a^{1.666666}$. $a^{\frac{11}{7}}=a^{1.57142}$.

These decimal indices form a very important class of numbers, called *logarithms*.

267. It is frequently convenient to vary the notation of powers of roots, by making use of a vinculum, or the radical sign $\sqrt{}$. In doing this, we must keep in mind, that the power of a root is the same as the root of a power; and also, that the *denominator* of a fractional exponent expresses a *root*, and the *numerator* a *power*.

Instead, therefore, of $a^{\frac{2}{3}}$, we may write $(a^{\frac{1}{3}})^2$, or $(a^2)^{\frac{1}{3}}$, or $\sqrt[3]{a^2}$.

The first of these three forms denotes the square of the cube root of a; and each of the two last, the cube root of the square of a.

So $a^{\frac{m}{n}} = \overline{a^{\frac{1}{n}}}\Big|^{m} = \overline{a^m}\Big|^{\frac{1}{n}} = \sqrt[n]{a^m}$.

And $(bx)^{\frac{3}{4}} = (b^3x^3)^{\frac{1}{4}} = \sqrt[4]{b^3x^3}$.

And $(a+y)^{\frac{3}{5}} = \overline{(a+y)^3}\Big|^{\frac{1}{5}} = \sqrt[5]{(a+y)^3}$.

268. Evolution is the opposite of involution. One is finding a *power* of a quantity, by multiplying it into itself. The other is finding a *root*, by resolving a quantity into equal factors. A quantity is resolved into any number of equal factors, by dividing its *index* into as many *equal parts*. (Art. 256.)

Evolution may be performed, then, by the following general rule;

Divide the index of the quantity by the number expressing the root to be found.

Or, place over the quantity the radical sign belonging to the required root.

1. Thus the cube root of a^6, is a^2. For $a^2 \times a^2 \times a^2 = a^6$.

Here 6, the index of the given quantity, is divided by 3, the number expressing the cube root.

2. The cube root of a or a^1, is $a^{\frac{1}{3}}$ or $\sqrt[3]{a}$.

For $a^{\frac{1}{3}} \times a^{\frac{1}{3}} \times a^{\frac{1}{3}}$, or $\sqrt[3]{a} \times \sqrt[3]{a} \times \sqrt[3]{a} = a$.

3. The 5th root of ab, is $(ab)^{\frac{1}{5}}$ or $\sqrt[5]{ab}$.

4. The nth root of a^2, is $a^{\frac{2}{n}}$ or $\sqrt[n]{a^2}$.

5. The 7th root of $2d-x$, is $(2d-x)^{\frac{1}{7}}$ or $\sqrt[7]{2d-x}$.

6. The 5th root of $(a-x)^3$, is $(a-x)^{\frac{3}{5}}$ or $\sqrt[5]{(a-x)^3}$.

7. The cube root of $a^{\frac{1}{2}}$, is $a^{\frac{1}{6}}$.

8. The 4th root of a^{-1}, is $a^{-\frac{1}{4}}$.

9. The cube root of $a^{\frac{2}{3}}$, is $a^{\frac{2}{9}}$.

10. The nth root of x^m, is $x^{\frac{m}{n}}$.

269. According to the rule just given, the cube root of the square root is found, by dividing the index $\frac{1}{2}$ by 3, as in

example 7th. But instead of dividing by 3, we may *multiply* by $\frac{1}{3}$. For $\frac{1}{2}\div3=\frac{1}{2}\div\frac{3}{1}=\frac{1}{2}\times\frac{1}{3}$. (Art. 165.)

So $\frac{1}{m}\div n=\frac{1}{m}\times\frac{1}{n}$. Therefore the mth root of the nth root of a is equal to $a^{\frac{1}{n}\times\frac{1}{m}}$.

That is, $\overline{a^{\frac{1}{n}}}|^{\frac{1}{m}}=a^{\frac{1}{n}\times\frac{1}{m}}=a^{\frac{1}{nm}}$.

Here the two fractional indices are reduced to one by multiplication.

It is sometimes necessary to *reverse* this process; to resolve an index into *two factors.*

Thus $x^{\frac{1}{8}}=x^{\frac{1}{4}\times\frac{1}{2}}=\overline{x^{\frac{1}{4}}}|^{\frac{1}{2}}$. That is, the 8th root of x is equal to the square root of the 4th root.

So $(a+b)^{\frac{1}{mn}}=(a+b)^{\frac{1}{m}\times\frac{1}{n}}=\overline{(a+b)^{\frac{1}{m}}}|^{\frac{1}{n}}$.

It may be necessary to observe, that resolving the *index* into factors, is not the same as resolving the *quantity* into factors. The latter is effected, by dividing the index into *parts.*

270. The rule in Art. 268, may be applied to every case in evolution. But when the quantity whose root is to be found, is composed of *several factors,* there will frequently be an advantage in taking the root of each of the factors *separately.*

This is done upon the principle that *the root of the product of several factors, is equal to the product of their roots.*

Thus $\sqrt{ab}=\sqrt{a}\times\sqrt{b}$. For each member of the equation if involved, will give the same power.

The square of $\sqrt{ab}$, is ab. (Art. 252.)

The square of $\sqrt{a}\times\sqrt{b}$, is $\sqrt{a}\times\sqrt{a}\times\sqrt{b}\times\sqrt{b}$.

But $\sqrt{a}\times\sqrt{a}=a$. And $\sqrt{b}\times\sqrt{b}=b$.

Therefore the square of $\sqrt{a}\times\sqrt{b}=\sqrt{a}\times\sqrt{a}\times\sqrt{b}\times\sqrt{b}$ $=ab$, which is also the square of $\sqrt{ab}$.

On the same principle, $(ab)^{\frac{1}{n}}=a^{\frac{1}{n}}b^{\frac{1}{n}}$.

When, therefore, a quantity consists of several factors, we may either extract the root of the whole together; or we may find the root of the factors separately, and then multiply them into each other.

Ex. 1. The cube root of xy, is either $(xy)^{\frac{1}{3}}$ or $x^{\frac{1}{3}}y^{\frac{1}{3}}$.

2. The 5th root of $3y$, is $\sqrt[5]{3y}$ or $\sqrt[5]{3}\times\sqrt[5]{y}$.

3. The 6th root of abh, is $(abh)^{\frac{1}{6}}$ or $a^{\frac{1}{6}}b^{\frac{1}{6}}h^{\frac{1}{6}}$.

4. The cube root of $8b$, is $(8b)^{\frac{1}{3}}$ or $2b^{\frac{1}{3}}$.

5. The nth root of x^ny, is $(x^ny)^{\frac{1}{n}}$ or $xy^{\frac{1}{n}}$.

271. *The root of a fraction is equal to the root of the numerator divided by the root of the denominator.*

1. Thus the square root of $\frac{a}{b}=\frac{a^{\frac{1}{2}}}{b^{\frac{1}{2}}}$. For $\frac{a^{\frac{1}{2}}}{b^{\frac{1}{2}}}\times\frac{a^{\frac{1}{2}}}{b^{\frac{1}{2}}}=\frac{a}{b}$.

2. So the nth root of $\frac{a}{b}=\frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}}$. For $\frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}}\times\frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}}\,..\,n$ times $=\frac{a}{b}$.

3. The square root of $\frac{x}{ay}$, is $\frac{\sqrt{x}}{\sqrt{ay}}$. 4. $\sqrt{\frac{ah}{xy}}=\frac{\sqrt{ah}}{\sqrt{xy}}$.

272. For determining what *sign* to prefix to a root, it is important to observe, that

An odd root of any quantity has the same sign as the quantity itself.

An even root of an affirmative quantity is ambiguous.

An even root of a negative quantity is impossible.

That the 3d, 5th, 7th, or any other *odd* root of a quantity must have the same sign as the quantity itself, is evident from Art. 233.

273. But an *even* root of an *affirmative* quantity may be either affirmative or negative. For, the quantity may be produced from the one, as well as from the other. (Art. 233.)

Thus the square root of a^2 is $+a$ or $-a$.

An even root of an affirmative quantity is, therefore, said to be *ambiguous*, and is marked with both $+$ and $-$.

Thus the square root of $3b$, is $\pm\sqrt{3b}$.

The 4th root of x, is $\pm x^{\frac{1}{4}}$

The ambiguity does not exist, however, when, from the nature of the case, or a previous multiplication, it is known whether the power has actually been produced from a positive or from a negative quantity.

274. But no even root of a *negative* quantity can be found.

The square root of $-a^2$ is neither $+a$ nor $-a$.

For $+a\times+a=+a^2$. And $-a\times-a=+a^2$ also.

An even root of a negative quantity is, therefore, said to be *impossible* or *imaginary*.

There are purposes to be answered, however, by applying the radical sign to negative quantities. The expression $\sqrt{-a}$ is often to be found in algebraic processes. For, although we are unable to assign it a rank, among either positive or negative quantities; yet we know that when multiplied into itself, its product is $-a$, because $\sqrt{-a}$ is by notation a root of $-a$, that is, a quantity which multiplied into itself produces $-a$.

This may, at first view, seem to be an exception to the general rule that the product of two negatives is affirmative. But it is to be considered, that $\sqrt{-a}$ is not itself a negative quantity, but the *root* of a negative quantity.

The mark of subtraction here, must not be confounded with that which is *prefixed* to the radical sign. The expression $\sqrt{-a}$ is not equivalent to $-\sqrt{a}$. The former is a root of $-a$; but the latter is a root of $+a$:

$$\text{For } -\sqrt{a}\times-\sqrt{a}=\sqrt{aa}=a.$$

The root of $-a$, however, may be *ambiguous*. It may be either $+\sqrt{-a}$, or $-\sqrt{-a}$.

One of the uses of imaginary expressions is to indicate an impossible or absurd supposition in the statement of a problem. Suppose it be required to divide the number 14 into two such parts, that their product shall be 60. If one of the parts be x, the other will be $14-x$. And by the supposition,

$$x\times(14-x)=60, \text{ or } 14x-x^2=60.$$

This reduced, by the rules in the following section, will give

$$x=7\pm\sqrt{-11}.$$

As the value of x is here found to contain an imaginary expression, we infer that there is an inconsistency in the statement of the problem: that the number 14 cannot be divided into any two parts whose product shall be 60.

275. Mathematicians are not entirely agreed in the logical explanations of imaginary quantities. It appears to be taken for granted, by Euler and others, that the *product* of the imaginary roots of two quantities is equal to the *root of the product* of the quantities; for instance, that

$$\sqrt{-a}\times\sqrt{-b}=\sqrt{-a\times-b}.$$

If this principle be admitted, certain limitations must be observed in the application. If we make

$$\sqrt{-a}\times\sqrt{-a}=\sqrt{-a\times-a},$$

and this, in conformity with the common rule for possible quantities, $=\sqrt{a^2}$; yet we are not at liberty to consider the latter expression as equivalent to a. For though $\sqrt{a^2}$, when taken without reference to its origin, is ambiguous, and may be either $+a$ or $-a$; yet when we know that it has been produced by multiplying $\sqrt{-a}$ into itself, we are not permitted to give it any other value than $-a$. (Art. 273.)

On the principle here stated, imaginary expressions may be easily prepared for calculation, by *resolving the quantity under the radical sign into two factors, one of which is* -1; thereby reducing the imaginary part of the expression to $\sqrt{-1}$. As $-a=+a\times-1$, the expression $\sqrt{-a}=\sqrt{a\times-1}=\sqrt{a}\times\sqrt{-1}$. So $\sqrt{-a-b}=\sqrt{a+b}\times\sqrt{-1}$. The first of the two factors is a real quantity. After the impossible part of imaginary expressions is thus reduced to $\sqrt{-1}$, they may be multiplied and divided by the rules already given for other radicals.

Thus in *Multiplication*,

1. $\sqrt{-a}\times\sqrt{-b}=\sqrt{a}\times\sqrt{-1}\times\sqrt{b}\times\sqrt{-1}=\sqrt{ab}\times-1=-\sqrt{ab}.$
2. $+\sqrt{-a}\times-\sqrt{-b}=-\sqrt{ab}\times-1=+\sqrt{ab}.$
3. $\sqrt{-9}\times\sqrt{-4}=-\sqrt{36}=-6.$
4. $(1+\sqrt{-1})\times(1-\sqrt{-1})=2.$

From these examples it will be seen, that according to the principle assumed, the product of two imaginary expressions is a real quantity.

5. $\sqrt{-a}\times\sqrt{b}=\sqrt{a}\times\sqrt{-1}\times\sqrt{b}=\sqrt{ab}\times\sqrt{-1}$.

6. $\sqrt{-2}\times\sqrt{18}=6\times\sqrt{-1}$.

Hence, the product of a real quantity and an imaginary expression, is itself imaginary.

In *Division*,

1. $\dfrac{\sqrt{-a}}{\sqrt{-b}}=\dfrac{\sqrt{a}\times\sqrt{-1}}{\sqrt{b}\times\sqrt{-1}}=\sqrt{\dfrac{a}{b}}$. 2. $\dfrac{\sqrt{-a}}{\sqrt{-a}}=1$.

Hence, the quotient of one imaginary expression divided by another is a real quantity.

3. $\dfrac{\sqrt{-a}}{\sqrt{b}}=\dfrac{\sqrt{a}\times\sqrt{-1}}{\sqrt{b}}=\sqrt{\dfrac{a}{b}}\times\sqrt{-1}$.

4. $\dfrac{\sqrt{a}}{\sqrt{-a}}=\dfrac{\sqrt{a}}{\sqrt{a}\times\sqrt{-1}}=\dfrac{1}{\sqrt{-1}}$.

Hence, the quotient of an imaginary quantity divided by a real one, or of a real quantity divided by an imaginary one, is itself imaginary.

By multiplying $\sqrt{-1}$ continually into itself, we obtain the following powers.

$(\sqrt{-1})^2=-1$	$(\sqrt{-1})^6=-1$
$(\sqrt{-1})^3=-\sqrt{-1}$	$(\sqrt{-1})^7=-\sqrt{-1}$
$(\sqrt{-1})^4=+1$	$(\sqrt{-1})^8=+1$
$(\sqrt{-1})^5=+\sqrt{-1}$	$(\sqrt{-1})^9=+\sqrt{-1}$
&c.	&c.

The even powers being alternately -1 and $+1$ and the odd powers, $-\sqrt{-1}$ and $+\sqrt{-1}$.

276. The methods of extracting the roots of *compound* quantities are to be considered in a future section. But there is one class of these, the squares of *binomial* and *residual* quantities, which it will be proper to attend to in this place. It has been shown (Art. 109,) that the square of a

binomial quantity consists of *three terms*, two of which are complete powers, and the other is a double product of the roots of these powers. The square of $a+b$, for instance, is

$$a^2+2ab+b^2,$$

two terms of which, a^2 and b^2, are complete powers, and $2ab$ is twice the product of a into b, that is, the root of a^2 into the root of b^2.

Whenever, therefore, we meet with a quantity of this description, we may know that its square root is a binomial; and this may be found, by taking the root of the two terms which are complete powers, and connecting them by the sign +. The other term disappears in the root. Thus, to find the square root of

$$x^2+2xy+y^2,$$

take the root of x^2, and the root of y^2, and connect them by the sign +. The binomial root will then be $x+y$.

In a *residual* quantity, the double product has the sign − prefixed, instead of +. The square of $a-b$, for instance, is $a^2-2ab+b^2$. (Art. 110.) And to obtain the root of a quantity of this description, we have only to take the roots of the two complete powers, and connect them by the sign −. Thus the square root of $x^2-2xy+y^2$ is $x-y$. Hence,

277. *To extract a binomial or residual square root, take the roots of the two terms which are complete powers, and connect them by the sign which is prefixed to the other term.*

Ex. 1. To find the root of x^2+2x+1.

The two terms which are complete powers are x^2 and 1.

The roots are x and 1. (Art. 259.)

The binomial root is, therefore, $x+1$.

2. The square root of x^2-2x+1, is $x-1$. (Art. 110.)
3. The square root of $a^2+a+\frac{1}{4}$, is $a+\frac{1}{2}$. (Art. 238.)
4. The square root of $a^2+\frac{4}{3}a+\frac{4}{9}$, is $a+\frac{2}{3}$.
5. The square root of $a^2+ab+\dfrac{b^2}{4}$, is $a+\dfrac{b}{2}$.
6. The square root of $a^2+\dfrac{2ab}{c}+\dfrac{b^2}{c^2}$, is $a+\dfrac{b}{c}$.

278. *A root whose value cannot be exactly expressed in numbers, is called a* SURD.

Thus $\sqrt{2}$ is a surd, because the square root of 2 cannot be expressed in numbers, with perfect exactness.

In decimals, it is 1.41421356 nearly.

But though we are unable to assign the value of such a quantity *when taken alone*, yet by multiplying it into itself, or by combining it with other quantities, we may produce expressions whose value can be determined. There is, therefore, a system of rules generally appropriated to surds. But as all quantities whatever, when under the same radical sign, or having the same index, may be treated in nearly the same manner; it will be most convenient to consider them together, under the general name of *Radical Quantities;* understanding by this term, every quantity which is found under a radical sign, or which has a fractional index.

279. Every quantity which is not a surd, is said to be *rational.* But for the purpose of distinguishing between radicals and other quantities, the term rational will be applied, in this section, to those only which do not appear under a radical sign, and which have not a fractional index.

REDUCTION OF RADICAL QUANTITIES.

280. Before entering on the consideration of the rules for the addition, subtraction, multiplication and division of radical quantities, it will be necessary to attend to the methods of reducing them from one form to another.

First, to reduce a *rational* quantity to the form of a radical;

Raise the quantity to a power of the same name as the given root, and then apply the corresponding radical sign or index.

Ex. 1. Reduce a to the form of the nth root.

The nth power of a is a^n. (Art. 228.)

Over this, place the radical sign, and it becomes $\sqrt[n]{a^n}$.

It is thus reduced to the form of a radical quantity, without any alteration of its value. For $\sqrt[n]{a^n}=a^{\frac{n}{n}}=a$.

2. Reduce 4 to the form of the cube root.

Ans. $\sqrt[3]{64}$ or $(64)^{\frac{1}{3}}$.

3. Reduce $3a$ to the form of the 4th root.

Ans. $\sqrt[4]{81a^4}$.

4. Reduce $\frac{1}{3}ab$ to the form of the square root.

Ans. $(\frac{1}{9}a^2b^2)^{\frac{1}{2}}$.

5. Reduce $3\times\overline{a-x}$ to the form of the cube root.

Ans. $\sqrt[3]{27\times(a-x)^3}$. See Art. 229.

6. Reduce a^2 to the form of the cube root.

The cube of a^2 is a^6.

And the cube root of a^6 is $\sqrt[3]{a^6}=\overline{a^6}|^{\frac{1}{3}}$.

In cases of this kind, where a *power* is to be reduced to the form of the nth root, it must be raised to the nth power, not of the *given letter*, but of the *power* of the letter.

Thus in the example, a^6 is the cube, not of a, but of a^2.

7. Reduce a^3b^4 to the form of the square root.

8. Reduce a^m to the form of the nth root.

281. *Secondly*, to reduce quantities which have different indices, to others of the same value having a *common index;*

1. Reduce the indices to a common denominator.

2. Involve each quantity to the power expressed by the numerator of its reduced index.

3. Take the root denoted by the common denominator.

Ex. 1. Reduce $a^{\frac{1}{4}}$ and $b^{\frac{1}{6}}$ to a common index.

1st. The indices $\frac{1}{4}$ and $\frac{1}{6}$ reduced to a common denominator, are $\frac{3}{12}$ and $\frac{2}{12}$. (Art. 150.)

2d. The quantities a and b involved to the powers expressed by the two numerators, are a^3 and b^2.

3d. The root denoted by the common denominator is $\frac{1}{12}$.

The answer, then, is $\overline{a^3}|^{\frac{1}{12}}$ and $\overline{b^2}|^{\frac{1}{12}}$.

The two quantities are thus reduced to a common index, without any alteration in their values.

For by Art. 265, $a^{\frac{1}{4}}=a^{\frac{3}{12}}$, which by Art. 269, $=\overline{a^3}|^{\frac{1}{12}}$.

And universally $a^{\frac{1}{n}}=a^{\frac{m}{mn}}=\overline{a^m}|^{\frac{1}{mn}}$.

2. Reduce $a^{\frac{1}{2}}$ and $bx^{\frac{2}{3}}$ to a common index.

The indices reduced to a common denominator are $\frac{3}{6}$ and $\frac{4}{6}$.

The quantities then, are $a^{\frac{3}{6}}$ and $(bx)^{\frac{4}{6}}$, or $\overline{a^3}|^{\frac{1}{6}}$, and $\overline{b^4x^4}|^{\frac{1}{6}}$.

3. Reduce a^2 and $b^{\frac{1}{n}}$. Ans. $\overline{a^{2n}}|^{\frac{1}{n}}$ and $b^{\frac{1}{n}}$.

4. Reduce $x^{\frac{1}{n}}$ and $y^{\frac{1}{m}}$. Ans. $\overline{x^m}|^{\frac{1}{mn}}$. And $\overline{y^n}|^{\frac{1}{mn}}$.

5. Reduce $2^{\frac{1}{2}}$ and $3^{\frac{1}{3}}$. Ans. $8^{\frac{1}{6}}$ and $9^{\frac{1}{6}}$.

6. Reduce $(a+b)^2$ and $(x-y)^{\frac{2}{3}}$.

Ans. $\overline{(a+b)^6}|^{\frac{1}{3}}$ and $\overline{(x-y)^2}|^{\frac{1}{3}}$.

7. Reduce $a^{\frac{1}{3}}$ and $b^{\frac{1}{5}}$. 8. Reduce $x^{\frac{2}{3}}$ and $5^{\frac{1}{2}}$.

282. When it is required to reduce a quantity to a *given* index;

Divide the index of the quantity by the given index, place the quotient over the quantity, and set the given index over the whole.

This is merely resolving the original index into two factors, according to Art. 269.

Ex. 1. Reduce $a^{\frac{1}{6}}$ to the index $\frac{1}{2}$.

By Art. 165, $\frac{1}{6}\div\frac{1}{2}=\frac{1}{6}\times\frac{2}{1}=\frac{2}{6}=\frac{1}{3}$.

This is the index to be placed over a, which then becomes $a^{\frac{1}{3}}$; and the given index set over this, makes it $\overline{a^{\frac{1}{3}}}|^{\frac{1}{2}}$, the answer.

2. Reduce a^2 and $x^{\frac{3}{2}}$ to the common index $\frac{1}{3}$.

$$\left.\begin{array}{l} 2\div\frac{1}{3}=2\times3=6, \text{ the first index} \\ \frac{3}{2}\div\frac{1}{3}=\frac{3}{2}\times3=\frac{9}{2}, \text{ the second index.} \end{array}\right\}$$

Therefore $(a^6)^{\frac{1}{3}}$ and $(x^{\frac{9}{2}})^{\frac{1}{3}}$ are the quantities required.

3. Reduce $4^{\frac{1}{2}}$ and $3^{\frac{1}{3}}$, to the common index $\frac{1}{6}$.

Ans. $(4^3)^{\frac{1}{6}}$ and $(3^2)^{\frac{1}{6}}$.

283. *Thirdly,* to remove a part of a root from under the radical sign;

If the quantity can be resolved into two factors, one of which is an exact power of the same name with the root;

find the root of this power, and prefix it to the other factor, with the radical sign between them.

This rule is founded on the principle, that the root of the *product* of two factors is equal to the product of their roots. (Art. 270.)

It will generally be best to resolve the radical quantity into such factors that one of them shall be the *greatest* power which will divide the quantity without a remainder. If there is no exact power which will divide the quantity, the reduction cannot be made.

Ex. 1. Remove a factor from $\sqrt{8}$.

The greatest square which will divide 8 is 4.

We may then resolve 8 into the factors 4 and 2. For $4\times2=8$.

The root of this product is equal to the product of the roots of its factors; that is, $\sqrt{8}=\sqrt{4}\times\sqrt{2}$.

But $\sqrt{4}=2$. Instead of $\sqrt{4}$, therefore, we may substitute its equal 2. We then have $2\times\sqrt{2}$ or $2\sqrt{2}$.

This is commonly called reducing a radical quantity to its *most simple terms.* But the learner may not perhaps at once perceive, that $2\sqrt{2}$ is a more simple expression than $\sqrt{8}$.

2. Reduce $\sqrt{a^2x}$. Ans. $\sqrt{a^2}\times\sqrt{x}=a\times\sqrt{x}=a\sqrt{x}$.

3. Reduce $\sqrt{18}$. Ans. $\sqrt{9\times2}=\sqrt{9}\times\sqrt{2}=3\sqrt{2}$.

4. Reduce $\sqrt[3]{64b^3c}$. Ans. $\sqrt[3]{64b^3}\times\sqrt[3]{c}=4b\sqrt[3]{c}$.

5. Reduce $\sqrt[4]{\dfrac{a^4b}{c^5d}}$. Ans. $\dfrac{a}{c}\sqrt[4]{\dfrac{b}{cd}}$. (Art. 271.)

6. Reduce $\sqrt[n]{a^nb}$. Ans. $a\sqrt[n]{b}$, or $ab^{\frac{1}{n}}$.

7. Reduce $(a^3-a^2b)^{\frac{1}{2}}$. Ans. $a(a-b)^{\frac{1}{2}}$.

8. Reduce $(54a^6b)^{\frac{1}{3}}$. Ans. $3a^2(2b)^{\frac{1}{3}}$.

9. Reduce $\sqrt{98a^2x}$. 10. Reduce $\sqrt[3]{a^3+a^3b^2}$.

284. By a contrary process, the co-efficient of a radical quantity may be introduced under the radical sign.

1. Thus, $a\sqrt[n]{b}=\sqrt[n]{a^nb}$.

For $a=\sqrt[n]{a^n}$ or $a^{\frac{n}{n}}$. (Art. 264.) And $\sqrt[n]{a^n}\times\sqrt[n]{b}=\sqrt[n]{a^nb}$.

Here the co-efficient a is first raised to a power of the same name as the radical part, and is then introduced as a factor under the radical sign.

2. $a(x-b)^{\frac{1}{3}}=(a^3\times\overline{x-b})^{\frac{1}{3}}=(a^3x-a^3b)^{\frac{1}{3}}$.

3. $2ab(2ab^2)^{\frac{1}{3}}=(16a^4b^5)^{\frac{1}{3}}$.

4. $\frac{a}{b}\left(\frac{b^2c}{a^2+b^2}\right)^{\frac{1}{2}}=\left(\frac{a^2b^2c}{a^2b^2+b^4}\right)^{\frac{1}{2}}$.

Addition and Subtraction of Radical Quantities.

285. Radical quantities may be added like rational quantities, *by writing them one after another with their signs.*

Thus the sum of $\sqrt{a}$ and $\sqrt{b}$, is $\sqrt{a}+\sqrt{b}$.

And the sum of $a^{\frac{1}{2}}-h^{\frac{1}{3}}$ and $x^{\frac{1}{4}}-y^{\frac{1}{n}}$, is $a^{\frac{1}{2}}-h^{\frac{1}{3}}+x^{\frac{1}{4}}-y^{\frac{1}{n}}$.

But in many cases, several terms may be reduced to one.

The sum of $2\sqrt{a}$ and $3\sqrt{a}$ is $2\sqrt{a}+3\sqrt{a}=5\sqrt{a}$.

For it is evident that twice the root of a, and three times the root of a, are five times the root of a. Hence,

286. When the quantities to be added have the same radical part, under the same radical sign or index; *add the rational parts, and to the sum annex the* RADICAL PARTS.

If no rational quantity is prefixed to the radical sign, 1 is always to be understood. (Art. 255.)

To	$2\sqrt[n]{ay}$	$5\sqrt{a}$	$3(x+h)^{\frac{1}{7}}$	$5bh^{\frac{1}{6}}$	$a\sqrt{b-h}$
Add	$\sqrt[n]{ay}$	$-2\sqrt{a}$	$4(x+h)^{\frac{1}{7}}$	$7bh^{\frac{1}{6}}$	$y\sqrt{b-h}$
Sum	$3\sqrt[n]{ay}$		$7(x+h)^{\frac{1}{7}}$		$(a+y)\times\sqrt{b-h}$

287. If the radical parts are originally different, they may sometimes be made alike, by the reductions in the preceding articles.

1. Add $\sqrt{8}$ to $\sqrt{50}$. Here the radical parts are not the same. But by the reduction in Art. 283, $\sqrt{8}=2\sqrt{2}$, and $\sqrt{50}=5\sqrt{2}$. The sum then is $7\sqrt{2}$.

2. Add $\sqrt{16b}$ to $\sqrt{4b}$. Ans. $4\sqrt{b}+2\sqrt{b}=6\sqrt{b}$.

3. Add $\sqrt{a^2x}$ to $\sqrt{b^4x}$. Ans. $a\sqrt{x}+b^2\sqrt{x}=(a+b^2)\times\sqrt{x}$.

4. Add $(36a^2y)^{\frac{1}{2}}$ to $(25y)^{\frac{1}{2}}$. Ans. $(6a+5)\times y^{\frac{1}{2}}$.

5. Add $\sqrt{18a}$ to $3\sqrt{2a}$.

288. But if the radical parts, after reduction, are *different* or have different *exponents*, they cannot be united in the same term; and must be added by writing them one after the other.

The sum of $3\sqrt{b}$ and $2\sqrt{a}$, is $3\sqrt{b}+2\sqrt{a}$.

It is manifest that three times the root of b, and twice the root of a, are neither five times the root of b, nor five times the root of a, unless b and a are equal.

The sum of $\sqrt[2]{a}$ and $\sqrt[3]{a}$, is $\sqrt[2]{a}+\sqrt[3]{a}$.

The *square* root of a, and the *cube* root of a, are neither twice the square root, nor twice the cube root of a.

289. *Subtraction* of radical quantities is to be performed in the same manner as addition, except that the signs in the subtrahend are to be changed according to Art. 75.

From	$\sqrt{ay}$	$4\sqrt[n]{a+x}$	$3h^{\frac{1}{3}}$	$a(x+y)$	$-a^{-\frac{1}{n}}$
Subtract	$3\sqrt{ay}$	$3\sqrt[n]{a+x}$	$-5h^{\frac{1}{3}}$	$b(x+y)$	$-2a^{-\frac{1}{n}}$
Difference	$-2\sqrt{ay}$		$8h^{\frac{1}{3}}$		$a^{-\frac{1}{n}}$

From $\sqrt{50}$, subtract $\sqrt{8}$. Ans. $5\sqrt{2}-2\sqrt{2}=3\sqrt{2}$.

From $\sqrt[3]{b^4y}$, subtract $\sqrt[3]{by^4}$. Ans. $(b-y)\times\sqrt[3]{by}$.

From $\sqrt[n]{x}$, subtract $\sqrt[5]{x}$.

Multiplication of Radical Quantities.

290. Radical quantities may be multiplied, like other quantities, by writing the factors one after another, either with or without the sign of multiplication between them.

Thus the product of $\sqrt{a}$ into $\sqrt{b}$, is $\sqrt{a}\times\sqrt{b}$.

The product of $h^{\frac{1}{3}}$ into $y^{\frac{1}{2}}$ is $h^{\frac{1}{3}}y^{\frac{1}{2}}$.

But it is often expedient to bring the factors under the same radical sign. This may be done, if they are first reduced to a common index.

Thus $\sqrt[n]{x}\times\sqrt[n]{y}=\sqrt[n]{xy}$. For the root of the product of several factors is equal to the product of their roots. Hence,

291. *Quantities under the same radical sign or index, may be multiplied together like rational quantities, the product being placed under the common radical sign or index.*

Multiply $\sqrt{x}$ into $\sqrt[3]{y}$, that is, $x^{\frac{1}{2}}$ into $y^{\frac{1}{3}}$.

The quantities reduced to the same index, (Art. 281,) are $(x^3)^{\frac{1}{6}}$, and $(y^2)^{\frac{1}{6}}$ and their product is $(x^3y^2)^{\frac{1}{6}}=\sqrt[6]{x^3y^2}$.

Multiply	$\sqrt{a+m}$	$\sqrt{dx}$	$a^{\frac{2}{3}}$	$(a+y)^{\frac{1}{n}}$	$a^{\frac{1}{m}}$
Into	$\sqrt{a-m}$	$\sqrt{hy}$	$x^{\frac{1}{3}}$	$(b+h)^{\frac{1}{n}}$	$x^{\frac{1}{n}}$
Product	$\sqrt{a^2-m^2}$		$(a^2x)^{\frac{1}{3}}$		$(a^nx^m)^{\frac{1}{mn}}$

Multiply $\sqrt{8xb}$ into $\sqrt{2xb}$. Prod. $\sqrt{16x^2b^2}=4xb$.

In this manner the product of radical quantities often becomes *rational.*

Thus the product of $\sqrt{2}$ into $\sqrt{18}=\sqrt{36}=6$.

And the product of $(a^2y^3)^{\frac{1}{4}}$ into $(a^2y)^{\frac{1}{4}}=(a^4y^4)^{\frac{1}{4}}=ay$.

292. *Roots of the same letter or quantity may be multiplied, by adding their fractional exponents.*

The exponents, like all other fractions, must be reduced to a common denominator, before they can be united in one term. (Art. 152.)

Thus $a^{\frac{1}{2}}\times a^{\frac{1}{3}}=a^{\frac{1}{2}+\frac{1}{3}}=a^{\frac{3}{6}+\frac{2}{6}}=a^{\frac{5}{6}}$

The values of the roots are not altered, by reducing their indices to a common denominator. (Art. 265.)

$$\left.\begin{array}{ll}\text{Therefore the first factor} & a^{\frac{1}{2}}=a^{\frac{3}{6}}\\ \text{And the second} & a^{\frac{1}{3}}=a^{\frac{2}{6}}\end{array}\right\}$$

But $a^{\frac{3}{6}}=a^{\frac{1}{6}}\times a^{\frac{1}{6}}\times a^{\frac{1}{6}}$. (Art. 261.)

And $a^{\frac{2}{6}}=a^{\frac{1}{6}}\times a^{\frac{1}{6}}$.

The product therefore is $a^{\frac{1}{6}}\times a^{\frac{1}{6}}\times a^{\frac{1}{6}}\times a^{\frac{1}{6}}\times a^{\frac{1}{6}}=a^{\frac{5}{6}}$.

And in all instances of this nature, the common denominator of the indices denotes a certain root; and the sum of the numerators, shows how often this is to be repeated as a factor to produce the required product.

Thus $a^{\frac{1}{n}}\times a^{\frac{1}{m}}=a^{\frac{m}{mn}}\times a^{\frac{n}{mn}}=a^{\frac{m+n}{mn}}$

Multiply	$3y^{\frac{1}{4}}$	$a^{\frac{1}{2}}\times a^{\frac{1}{3}}$	$(a+b)^{\frac{1}{2}}$	$(a-y)^{\frac{1}{n}}$	$x^{-\frac{1}{4}}$
Into	$y^{\frac{2}{3}}$	$a^{\frac{1}{4}}$	$(a+b)^{\frac{1}{4}}$	$(a-y)^{\frac{1}{m}}$	$x^{-\frac{1}{3}}$
Product	$3y^{\frac{11}{12}}$		$(a+b)^{\frac{3}{4}}$		$x^{-\frac{7}{12}}$

The product of $y^{\frac{1}{2}}$ into $y^{-\frac{1}{3}}$, is $y^{\frac{3}{6}-\frac{2}{6}}=y^{\frac{1}{6}}$.

The product of $a^{\frac{1}{n}}$ into $a^{-\frac{1}{n}}$, is $a^{\frac{1}{n}-\frac{1}{n}}=a^{0}=1$.

And $x^{n-\frac{1}{2}}\times x^{\frac{1}{2}-n}=x^{n-n+\frac{1}{2}-\frac{1}{2}}=x^{0}=1$.

The product of a^{2} into $a^{\frac{1}{3}}=a^{\frac{6}{3}}\times a^{\frac{1}{3}}=a^{\frac{7}{3}}$.

293. From the last example it will be seen, that *powers* and *roots* may be multiplied by a common rule. This is one of the many advantages derived from the notation by fractional indices. Any quantities whatever may be reduced to the form of radicals, (Art. 280,) and may then be subjected to the same modes of operation.

Thus $y^{3}\times y^{\frac{1}{6}}=y^{3+\frac{1}{6}}=y^{\frac{19}{6}}$.

And $x\times x^{\frac{1}{n}}=x^{1+\frac{1}{n}}=x^{\frac{n+1}{n}}$.

The product will become rational, whenever the numerator of the index can be exactly divided by the denominator.

Thus $a^{3}\times a^{\frac{1}{3}}\times a^{\frac{2}{3}}=a^{\frac{12}{3}}=a^{4}$.

And $(a+b)^{\frac{4}{3}}\times(a+b)^{-\frac{1}{3}}=(a+b)^{\frac{3}{3}}=a+b$.

And $a^{\frac{3}{5}}\times a^{\frac{2}{5}}=a^{\frac{5}{5}}=a$.

294. When radical quantities which are reduced to the same index, have *rational co-efficients, the rational parts may be multiplied together, and their product prefixed to the product of the radical parts.*

1. Multiply $a\sqrt{b}$ into $c\sqrt{d}$.

The product of the rational parts is ac.

The product of the radical parts is $\sqrt{bd}$.

And the whole product is $ac\sqrt{bd}$.

For $a\sqrt{b}$ is $a\times\sqrt{b}$. (Art. 255.) And $c\sqrt{d}$ is $c\times\sqrt{d}$.

By Art. 96, $a\times\sqrt{b}$ into $c\times\sqrt{d}$, is $a\times\sqrt{b}\times c\times\sqrt{d}$; or by changing the order of the factors,

$$a\times c\times\sqrt{b}\times\sqrt{d}=ac\times\sqrt{bd}=ac\sqrt{bd}.$$

2. Multiply $ax^{\frac{1}{2}}$ into $bd^{\frac{1}{3}}$.

When the radical parts are reduced to a common index, the factors become $a(x^3)^{\frac{1}{6}}$ and $b(d^2)^{\frac{1}{6}}$.

The product then is $ab(x^3d^2)^{\frac{1}{6}}$.

But in cases of this nature we may save the trouble of reducing to a common index, by multiplying as in Art. 290.

Thus $ax^{\frac{1}{2}}$ into $bd^{\frac{1}{3}}$, is $ax^{\frac{1}{2}}bd^{\frac{1}{3}}$.

Multiply	$a(b+x)^{\frac{1}{2}}$	$a\sqrt{y^2}$	$a\sqrt{x}$	$ax^{-\frac{1}{2}}$	$x\sqrt[3]{3}$
Into	$y(b-x)^{\frac{1}{2}}$	$b\sqrt{hy}$	$b\sqrt{x}$	$by^{-\frac{1}{2}}$	$y\sqrt[3]{9}$
Product	$ay(b^2-x^2)^{\frac{1}{2}}$		$ab\sqrt{x^2}=abx$		$3xy$

295. If the rational quantities, instead of being *co-efficients* to the radical quantities, are connected with them by the signs + and −, each term in the multiplier must be multiplied into each in the multiplicand, as in Art. 95.

$$\begin{array}{ll} \text{Multiply} & a+\sqrt{b} \\ \text{Into} & c+\sqrt{d} \\ \hline & ac+c\sqrt{b} \\ & \qquad a\sqrt{d}+\sqrt{bd} \\ \hline & ac+c\sqrt{b}+a\sqrt{d}+\sqrt{bd}. \end{array}$$

The product of $a+\sqrt{y}$ into $1+r\sqrt{y}$, is $a+\sqrt{y}+ar\sqrt{y}+ry$.

1. Multiply $\sqrt{a}$ into $\sqrt[3]{b}$. Ans. $\sqrt[6]{a^3b^2}$.
2. Multiply $5\sqrt{5}$ into $3\sqrt{8}$. Ans. $30\sqrt{10}$.
3. Multiply $2\sqrt{3}$ into $3\sqrt[3]{4}$. Ans. $6\sqrt[6]{432}$.
4. Multiply $\sqrt{d}$ into $\sqrt[3]{ab}$. Ans. $\sqrt[6]{a^2b^2d^3}$.
5. Multiply $\sqrt{\frac{2ab}{3c}}$ into $\sqrt{\frac{9ad}{2b}}$. Ans. $\sqrt{\frac{3a^2d}{c}}$.
6. Multiply $a(a-x)^{\frac{1}{2}}$ into $(c-d)\times(ax)^{\frac{1}{2}}$.
   Ans. $(ac-ad)\times(a^2x-ax^2)^{\frac{1}{2}}$.

Division of Radical Quantities.

296. The division of radical quantities may be expressed by writing the divisor under the dividend, in the form of a fraction.

Thus the quotient of $\sqrt[3]{a}$ divided by $\sqrt{b}$, is $\frac{\sqrt[3]{a}}{\sqrt{b}}$.

And $(a+h)^{\frac{1}{3}}$ divided by $(b+x)^{\frac{1}{n}}$, is $\frac{(a+h)^{\frac{1}{3}}}{(b+x)^{\frac{1}{n}}}$.

In these instances, the radical sign or index is *separately* applied to the numerator and the denominator. But if the divisor and dividend are reduced to the *same* index or radical sign, this may be applied to the *whole* quotient.

Thus $\sqrt[n]{a}\div\sqrt[n]{b}=\frac{\sqrt[n]{a}}{\sqrt[n]{b}}=\sqrt[n]{\frac{a}{b}}$. For the root of a fraction is equal to the root of the numerator divided by the root of the denominator. (Art. 271.)

Again, $\sqrt[n]{ab}\div\sqrt[n]{b}=\sqrt[n]{a}$. For the product of this quotient into the divisor is equal to the dividend, that is,

$$\sqrt[n]{a}\times\sqrt[n]{b}=\sqrt[n]{ab}.\quad\text{Hence,}$$

297. *Quantities under the same radical sign or index may be divided like rational quantities, the quotient being placed under the common radical sign or index.*

Divide $(x^3y^2)^{\frac{1}{6}}$ by $y^{\frac{1}{3}}$.

These reduced to the same index are $(x^3y^2)^{\frac{1}{6}}$ and $(y^2)^{\frac{1}{6}}$:

And the quotient is $(x^3)^{\frac{1}{6}}=x^{\frac{3}{6}}=x^{\frac{1}{2}}$.

Divide	$\sqrt{6a^3x}$	$\sqrt{dhx^2}$	$(a^3+ax)^{\frac{1}{3}}$	$(a^3h)^{\frac{1}{m}}$	$(a^2y^2)^{\frac{1}{4}}$
By	$\sqrt{3x}$	$\sqrt{dx}$	$a^{\frac{1}{3}}$	$(ax)^{\frac{1}{m}}$	$(ay)^{\frac{1}{4}}$
Quotient	$\sqrt{2a^3}$		$(a^2+x)^{\frac{1}{3}}$		$(ay)^{\frac{1}{4}}$.

298. *A root is divided by another root of the same letter or quantity, by subtracting the index of the divisor from that of the dividend.*

Thus $a^{\frac{1}{2}}\div a^{\frac{1}{6}}=a^{\frac{1}{2}-\frac{1}{6}}=a^{\frac{3}{6}-\frac{1}{6}}=a^{\frac{2}{6}}=a^{\frac{1}{3}}$.

For $a^{\frac{1}{2}}=a^{\frac{3}{6}}=a^{\frac{1}{6}}\times a^{\frac{1}{6}}\times a^{\frac{1}{6}}$ and this divided by $a^{\frac{1}{6}}$ is

$$\frac{a^{\frac{1}{6}}\times a^{\frac{1}{6}}\times a^{\frac{1}{6}}}{a^{\frac{1}{6}}}=a^{\frac{1}{6}}\times a^{\frac{1}{6}}=a^{\frac{2}{6}}=a^{\frac{1}{3}}.$$

In the same manner, it may be shown that $a^{\frac{1}{m}}\div a^{\frac{1}{n}}=a^{\frac{1}{m}-\frac{1}{n}}$.

Divide	$(3a)^{\frac{1}{1}\frac{1}{2}}$	$(ax)^{\frac{2}{3}}$	$a^{\frac{m+n}{mn}}$	$(b+y)^{\frac{2}{n}}$	$(r^2y^3)^{\frac{1}{7}}$
By	$(3a)^{\frac{2}{3}}$	$(ax)^{\frac{1}{3}}$	$a^{\frac{1}{m}}$	$(b+y)^{\frac{1}{n}}$	$(r^2y^3)^{\frac{3}{7}}$
Quotient	$(3a)^{\frac{1}{4}}$		$a^{\frac{1}{n}}$		$(r^2y^3)^{-\frac{2}{7}}$

Powers and *roots* may be brought promiscuously together, and divided according to the same rule. See Art. 293.

Thus $a^2\div a^{\frac{1}{3}}=a^{2-\frac{1}{3}}=a^{\frac{5}{3}}$. For $a^{\frac{5}{3}}\times a^{\frac{1}{3}}=a^{\frac{6}{3}}=a^2$.

So $y^n\div y^{\frac{1}{m}}=y^{n-\frac{1}{m}}$.

299. When radical quantities which are reduced to the same index have *rational co-efficients, the rational parts may be divided separately, and their quotient prefixed to the quotient of the radical parts.*

Thus $ac\sqrt{bd}\div a\sqrt{b}=c\sqrt{d}$. For this quotient multiplied into the divisor is equal to the dividend.

Divide	$24x\sqrt{ay}$	$18dh\sqrt{bx}$	$by(a^3x^2)^{\frac{1}{n}}$	$16\sqrt{32}$	$b\sqrt{xy}$
By	$6\sqrt{a}$	$2h\sqrt{x}$	$y(a\ x\ )^{\frac{1}{n}}$	$8\sqrt{4}$	$\sqrt{y}$
Quotient	$4x\sqrt{y}$		$b(a^2x\ )^{\frac{1}{n}}$		$b\sqrt{x}$

Divide $ab(x^2b)^{\frac{1}{4}}$ by $a(x)^{\frac{1}{2}}$.

These reduced to the same index are $ab(x^2b)^{\frac{1}{4}}$ and $a(x^2)^{\frac{1}{4}}$.

The quotient then is $b(b)^{\frac{1}{4}}=(b^5)^{\frac{1}{4}}$. (Art. 284.)

To save the trouble of reducing to a common index, the division may be expressed in the form of a fraction.

The quotient will then be $\dfrac{ab(x^2b)^{\frac{1}{4}}}{a(x)^{\frac{1}{2}}}$.

1. Divide $2\sqrt[3]{bc}$ by $3\sqrt{ac}$. Ans. $\frac{2}{3}\sqrt[6]{\dfrac{b^2}{a^3c}}$.
2. Divide $10\sqrt[3]{108}$ by $5\sqrt[3]{4}$. Ans. $2\sqrt[3]{27}=6$.
3. Divide $10\sqrt{27}$ by $2\sqrt{3}$. Ans. 15.
4. Divide $8\sqrt{108}$ by $2\sqrt{6}$. Ans. $12\sqrt{2}$.
5. Divide $(a^2b^2d^3)^{\frac{1}{6}}$ by $d^{\frac{1}{2}}$. Ans. $(ab)^{\frac{1}{3}}$.
6. Divide $(16a^3-12a^2x)^{\frac{1}{2}}$ by $2a$. Ans. $(4a-3x)^{\frac{1}{2}}$.

Involution of Radical Quantities.

300. *Radical quantities, like powers, are involved by multiplying the index of the root into the index of the required power.*

1. The square of $a^{\frac{1}{3}}=a^{\frac{1}{3}\times 2}=a^{\frac{2}{3}}$. For $a^{\frac{1}{3}}\times a^{\frac{1}{3}}=a^{\frac{2}{3}}$.
2. The cube of $a^{\frac{1}{4}}=a^{\frac{1}{4}\times 3}=a^{\frac{3}{4}}$. For $a^{\frac{1}{4}}\times a^{\frac{1}{4}}\times a^{\frac{1}{4}}=a^{\frac{3}{4}}$.
3. And universally, the nth power of $a^{\frac{1}{m}}=a^{\frac{1}{m}\times n}=a^{\frac{n}{m}}$.

For the nth power of $a^{\frac{1}{m}}=a^{\frac{1}{m}}\times a^{\frac{1}{m}}\ldots.n$ times, and the sum of the indices will then be $\frac{n}{m}$.

4. The 5th power of $a^{\frac{1}{2}}y^{\frac{1}{3}}$, is $a^{\frac{5}{2}}y^{\frac{5}{3}}$. Or, by reducing the roots to a common index,

$$(a^3y^2)^{\frac{1}{6}\times 5}=(a^3y^2)^{\frac{5}{6}}.$$

5. The cube of $a^{\frac{1}{n}}x^{\frac{1}{m}}$, is $a^{\frac{3}{n}}x^{\frac{3}{m}}$ or $(a^mx^n)^{\frac{3}{mn}}$.

6. The square of $a^{\frac{2}{3}}x^{\frac{3}{4}}$, is $a^{\frac{4}{3}}x^{\frac{6}{4}}$.

The cube of $a^{\frac{1}{3}}$, is $a^{\frac{1}{3}\times 3}=a^{\frac{3}{3}}=a$.

And the nth power of $a^{\frac{1}{n}}$, is $a^{\frac{n}{n}}=a$. That is,

301. *A root is raised to a power of the same name, by removing the index or radical sign.*

Thus the cube of $\sqrt[3]{b+x}$, is $b+x$.

And the nth power of $(a-y)^{\frac{1}{n}}$, is $(a-y)$.

302. When the radical quantities have *rational co-efficients*, these must also be involved.

1. The square of $a\sqrt[n]{x}$, is $a^2\sqrt[n]{x^2}$.

For $a\sqrt[n]{x}\times a\sqrt[n]{x}=a^2\sqrt[n]{x^2}$.

2. The nth power of $a^mx^{\frac{1}{m}}$, is $a^{mn}x^{\frac{n}{m}}$.

3. The square of $a\sqrt{x-y}$, is $a^2\times(x-y)$.

4. The cube of $3a\sqrt[3]{y}$, is $27a^3y$.

303. But if the radical quantities are connected with others by the signs $+$ and $-$, they must be involved by a multiplication of the several terms, as in Art. 230.

Ex. 1. Required the squares of $a+\sqrt{y}$ and $a-\sqrt{y}$.

$$\begin{array}{ll}
a+\sqrt{y} & \qquad a-\sqrt{y} \\
a+\sqrt{y} & \qquad a-\sqrt{y} \\
\hline
a^2+a\sqrt{y} & \qquad a^2-a\sqrt{y} \\
\qquad a\sqrt{y}+y & \qquad\quad -a\sqrt{y}+y \\
\hline
a^2+2a\sqrt{y}+y & \qquad a^2-2a\sqrt{y}+y
\end{array}$$

2. Required the cube of $a-\sqrt{b}$.

3. Required the cube of $2d+\sqrt{x}$.

304. It is unnecessary to give a separate rule for the *evolution* of radical quantities, that is, for finding the root of a quantity which is already a root. The operation is the same as in other cases of evolution. The fractional index of the radical quantity is to be divided, by the number expressing the root to be found. Or, the radical sign belonging to the required root, may be placed over the given quantity. (Art. 268.) If there are rational co-efficients, the roots of these must also be extracted.

Thus, the square root of $a^{\frac{1}{3}}$, is $a^{\frac{1}{3}\div 2}=a^{\frac{1}{6}}$.

The cube root of $a(xy)^{\frac{1}{3}}$, is $a^{\frac{1}{3}}(xy)^{\frac{1}{9}}$.

The nth root of $a\sqrt[2]{by}$, is $\sqrt[n]{a\sqrt[2]{by}}$

305. It may be proper to observe, that dividing the *fractional* index of a root is the same in effect, as *multiplying* the number which is placed over the radical sign. For this number corresponds with the *denominator* of the fractional index; and a fraction is divided, by *multiplying* its denominator.

Thus $\sqrt[2]{a}=a^{\frac{1}{2}}$ $\qquad$ $\sqrt[6]{a}=a^{\frac{1}{6}}$.

$\sqrt[4]{a}=a^{\frac{1}{4}}$ $\qquad$ $\sqrt[2n]{a}=a^{\frac{1}{2n}}$.

On the other hand, *multiplying* the fractional index is equivalent to *dividing* the number which is placed over the radical sign.

Thus the square of $\sqrt[6]{a}$ or $a^{\frac{1}{6}}$, is $\sqrt[3]{a}$ or $a^{\frac{1}{6}\times 2}=a^{\frac{1}{3}}$.

306. In algebraic calculations, we have sometimes occasion to seek for a factor, which multiplied into a given radical quantity, will render the product *rational*. In the case of a *simple* radical, such a factor is easily found. For if the nth root of any quantity, be multiplied by the same root raised to a power whose index is $n-1$, the product will be the given quantity.

Thus $\sqrt[n]{x}\times\sqrt[n]{x^{n-1}}$ or $x^{\frac{1}{n}}\times x^{\frac{n-1}{n}}=x^{\frac{n}{n}}=x$.

And $(x+y)^{\frac{1}{n}}\times(x+y)^{\frac{n-1}{n}}=x+y$.

So $\sqrt{a}\times\sqrt{a}=a$. And $\sqrt[3]{a}\times\sqrt[3]{a^2}=\sqrt[3]{a^3}=a$.

And $\sqrt[3]{a}\times\sqrt[3]{a^2}=a$, &c. And $(a+b)^{\frac{1}{3}}\times(a+b)^{\frac{2}{3}}=a+b$.

And $(x+y)^{\frac{1}{4}}\times(x+y)^{\frac{3}{4}}=x+y$.

307. A factor which will produce a rational product, when multiplied into a *binomial surd* containing only the *square root*, may be found by applying the principle, that the product of the sum and difference of two quantities, is equal to the difference of their squares. (Art. 111.) The binomial itself, after the sign which connects the terms is changed from + to −, or from − to +, will be the factor required.

Thus $(\sqrt{a}+\sqrt{b})\times(\sqrt{a}-\sqrt{b})=\sqrt{a^2}-\sqrt{b^2}=a-b$, which is free from radicals.

So $(1+\sqrt{2})\times(1-\sqrt{2})=1-2=-1$.

And $(3-2\sqrt{2})\times(3+2\sqrt{2})=1$.

When the compound surd consists of *more than two* terms, it may be reduced, by successive multiplications, first to a binomial surd, and then to a rational quantity.

Thus $(\sqrt{10}-\sqrt{2}-\sqrt{3})\times(\sqrt{10}+\sqrt{2}+\sqrt{3})=5-2\sqrt{6}$, a binomial surd.

And $(5-2\sqrt{6})\times(5+2\sqrt{6}=1$.

Therefore $(\sqrt{10}-\sqrt{2}-\sqrt{3})$ multiplied into $(\sqrt{10}+\sqrt{2}+\sqrt{3})\times(5+2\sqrt{6})=1$.

308. It is sometimes desirable to clear from radical signs the numerator or denominator of a *fraction*. This may be effected, without altering the value of the fraction, if the numerator and denominator be both multiplied by a factor which will render either of them rational, as the case may require.

1. If both parts of the fraction $\frac{\sqrt{a}}{\sqrt{x}}$ be multiplied by $\sqrt{a}$, it will become $\frac{\sqrt{a}\times\sqrt{a}}{\sqrt{x}\times\sqrt{a}}=\frac{a}{\sqrt{ax}}$, in which the *numerator* is a rational quantity.

Or if both parts of the given fraction be multiplied by $\sqrt{x}$, it will become $\frac{\sqrt{ax}}{x}$, in which the *denominator* is rational.

2. The fraction $\dfrac{b^{\frac{1}{2}}}{(a+x)^{\frac{1}{3}}}=\dfrac{b^{\frac{1}{2}}\times(a+x)^{\frac{2}{3}}}{(a+x)^{\frac{1}{3}+\frac{2}{3}}}=\dfrac{b^{\frac{1}{2}}\times(a+x)^{\frac{2}{3}}}{a+x}$.

3. The fraction $\dfrac{\sqrt[3]{y+x}}{a}=\dfrac{(y+x)^{\frac{1}{3}+\frac{2}{3}}}{a(y+x)^{\frac{2}{3}}}=\dfrac{y+x}{a(y+x)^{\frac{2}{3}}}$.

4. The fraction $\dfrac{a}{x^{\frac{1}{n}}}=\dfrac{ax^{\frac{n-1}{n}}}{x^{\frac{1}{n}}\times x^{\frac{n-1}{n}}}=\dfrac{a\sqrt[n]{x^{n-1}}}{x}$.

5. The fraction $\dfrac{\sqrt{2}}{3-\sqrt{2}}=\dfrac{\sqrt{2}\times(3+\sqrt{2})}{(3-\sqrt{2})(3+\sqrt{2})}=\dfrac{2+3\sqrt{2}}{7}$.

6. The fraction $\dfrac{3}{\sqrt{5}-\sqrt{2}}=\dfrac{3(\sqrt{5}+\sqrt{2})}{(\sqrt{5}-\sqrt{2})(\sqrt{5}+\sqrt{2})}=\sqrt{5}+\sqrt{2}$.

7. The fraction $\dfrac{6}{5^{\frac{1}{4}}}=\dfrac{6\times5^{\frac{3}{4}}}{5^{\frac{1}{4}+\frac{3}{4}}}=\dfrac{6}{5}\sqrt[4]{125}$.

8. The fraction

$$\frac{8}{\sqrt{3}+\sqrt{2}+1}=\frac{8\times(\sqrt{3}-\sqrt{2}-1)(-\sqrt{2})}{(\sqrt{3}+\sqrt{2}+1)(\sqrt{3}-\sqrt{2}-1)(-\sqrt{2})}$$
$$=4-2\sqrt{6}+2\sqrt{2}.$$

9. Reduce $\dfrac{2}{\sqrt{3}}$ to a fraction having a rational denominator.

10. Reduce $\dfrac{a-\sqrt{b}}{a+\sqrt{b}}$ to a fraction having a rational denominator.

309. In a similar manner, *an equation* may be cleared of radicals by involution, or by multiplying both sides by such factors as will render the radical quantities rational.

Ex. 1. Clear from its radical sign the equation

$$\sqrt{ac+d}=x-ad.$$

Squaring both sides,

$$ac+d=x^2-2adx+a^2d^2.$$

2. Clear from radicals the equation

$$\sqrt{b+a}-\sqrt{a+x}=y.$$

Transposing and squaring,

$$b+a=y^2+2y\sqrt{a+x}+a+x.$$

Transposing and squaring again,

$$(b-y^2-x)^2=4y^2\times(a+x).$$

310. The arithmetical operation of finding the proximate value of a fractional surd, may be shortened, by rendering either the numerator or the denominator rational. The root of a fraction is equal to the root of the numerator divided by the root of the denominator. (Art. 271.)

Thus $\sqrt[n]{\frac{a}{b}}=\frac{\sqrt[n]{a}}{\sqrt[n]{b}}$. But this may be reduced to $\frac{a}{\sqrt[n]{b}\times\sqrt[n]{a^{n-1}}}$ or $\frac{\sqrt[n]{a}\times\sqrt[n]{b^{n-1}}}{b}$. (Art. 308.)

The square root of $\frac{a}{b}$ is $\frac{\sqrt{a}}{\sqrt{b}}$, or $\frac{a}{\sqrt{ab}}$, or $\frac{\sqrt{ab}}{b}$.

When the fraction is thrown into this form, the process of extracting the root arithmetically, will be confined either to the numerator, or to the denominator.

Thus the square root of $\frac{3}{7}=\frac{\sqrt{3}}{\sqrt{7}}=\frac{\sqrt{3}\times\sqrt{7}}{\sqrt{7}\times\sqrt{7}}=\frac{\sqrt{21}}{7}$.

Examples.

1. Find the 4th root of $81a^2$.
2. Find the 6th root of $(a+b)^{-3}$.
3. Fnd the nth root of $(x-y)^{\frac{1}{3}}$.
4. Find the cube root of $-125ax^6$.
5. Find the square root of $\frac{4a^4}{9x^2y^2}$.
6. Find the 5th root of $\frac{32a^5x^{10}}{243}$.
7. Find the square root of $x^2-6bx+9b^2$.
8. Find the square root of $a^2+ay+\frac{y^2}{4}$.
9. Reduce ax^2 to the form of the 6th root.

10. Reduce $-3y$ to the form of the cube root.

11. Reduce a^2 and $a^{\frac{1}{3}}$ to a common index.

12. Reduce $4^{\frac{1}{3}}$ and $5^{\frac{1}{4}}$ to a common index.

13. Reduce $a^{\frac{1}{2}}$ and $b^{\frac{1}{4}}$ to the common index $\frac{1}{8}$.

14. Reduce $2^{\frac{1}{2}}$ and $4^{\frac{1}{4}}$ to the common index $\frac{1}{8}$.

15. Remove a factor from $\sqrt{294}$.

16. Remove a factor from $\sqrt{x^3-a^2x^2}$.

17. Find the sum and difference of $\sqrt{16a^2x}$ and $\sqrt{4a^2x}$.

18. Find the sum and difference of $\sqrt[3]{192}$ and $\sqrt[3]{24}$.

19. Multiply $7\sqrt[3]{18}$ into $5\sqrt[3]{4}$.

20. Multiply $4+2\sqrt{2}$ into $2-\sqrt{2}$.

21. Multiply $a(a+\sqrt{c})^{\frac{1}{2}}$ into $b(a-\sqrt{c})^{\frac{1}{2}}$.

22. Multiply $2(a+b)^{\frac{1}{n}}$ into $3(a+b)^{\frac{1}{m}}$.

23. Divide $6\sqrt{54}$ by $3\sqrt{2}$.

24. Divide $4\sqrt[3]{72}$ by $2\sqrt[3]{18}$.

25. Divide $\sqrt{7}$ by $\sqrt[3]{7}$.

26. Divide $8\sqrt[3]{512}$ by $4\sqrt[3]{2}$.

27. Find the cube of $17\sqrt{21}$.

28. Find the square of $5+\sqrt{2}$.

29. Find the 4th power of $\frac{1}{6}\sqrt{6}$.

30. Find the cube of $\sqrt{x}-\sqrt{b}$.

31. Find a factor which will make $\sqrt[3]{y}$ rational.

32. Find a factor which will make $\sqrt{5}-\sqrt{x}$ rational.

33. Reduce $\dfrac{\sqrt{a}}{\sqrt{x}}$ to a fraction having a rational numerator.

34. Reduce $\dfrac{\sqrt{6}}{\sqrt{7}\times\sqrt{3}}$ to a fraction having a rational denominator.

SECTION X.

REDUCTION OF EQUATIONS BY INVOLUTION AND EVOLUTION.

Art. **311.** In an equation, the letter which expresses the unknown quantity is sometimes found under a *radical sign*. We may have $\sqrt{x}=a.$

To clear this of the radical sign, let each member of the equation be squared, that is, multiplied into itself. We shall then have

$\sqrt{x}\times\sqrt{x}=aa.$ Or, (Art. 301,) $x=a^2.$

The equality of the sides is not affected by this operation, because each is only multiplied into itself, that is, equal quantities are multiplied into equal quantities.

The same principle is applicable to any root whatever. If $\sqrt[n]{x}=a$; then $x=a^n$. For by Art. 301, a root is raised to a power of the same name, by removing the index or radical sign. Hence,

312. *When the unknown quantity is under a radical sign, the equation is reduced by involving both sides,* to a power of the same name, as the root expressed by the radical sign.

It will generally be expedient to make the necessary transpositions, *before* involving the quantities; so that all those which are not under the radical sign may stand on one side of the equation.

Ex. 1. Reduce the equation $\sqrt{x}+4=9$
Transposing $+4$ $\sqrt{x}=9-4=5$
Involving both sides, $x=5^2=25.$

2. Reduce the equation $a+\sqrt[n]{x}-b=d$
By transposition, $\sqrt[n]{x}=d+b-a$
By involution, $x=(d+b-a)^n.$

3. Reduce the equation $\sqrt[3]{x+1}=4$
Involving both sides, $x+1=4^3=64$
And $x=63$

4. Reduce the equation $4+3\sqrt{x-4}=6+\frac{1}{2}$
Clearing of fractions, $8+6\sqrt{x-4}=13$
And $\sqrt{x-4}=\frac{5}{6}$.
Involving both sides, $x-4=\frac{25}{36}$
And $x=\frac{25}{36}+4$.

5. Reduce the equation $\sqrt{a^2+\sqrt{x}}=\dfrac{3+d}{\sqrt{a^2+\sqrt{x}}}$
Multiplying by $\sqrt{a^2+\sqrt{x}}$, $a^2+\sqrt{x}=3+d$
And $\sqrt{x}=3+d-a^2$
Involving both sides, $x=(3+d-a^2)^2$.

In the first step in this example, multiplying the first member into $\sqrt{a^2+\sqrt{x}}$, that is, into itself, is the same as squaring it, which is done by taking away its radical sign. The other member being a fraction, is multiplied into a quantity equal to its denominator, by cancelling the denominator. (Art. 162.) There remains a radical sign over x, which must be removed by involving both sides of the equation.

6. Reduce $3+2\sqrt{x}-\frac{4}{5}=6$. Ans. $x=\frac{361}{100}$.

7. Reduce $4\sqrt{\dfrac{x}{5}}=8$. Ans. $x=20$.

8. Reduce $(2x+3)^{\frac{1}{3}}+4=7$. Ans. $x=12$.

9. Reduce $\sqrt{12+x}=2+\sqrt{x}$. Ans. $x=4$.

10. Reduce $\sqrt{x-a}=\sqrt{x}-\frac{1}{2}\sqrt{a}$. Ans. $x=\dfrac{25a}{16}$.

11. Reduce $\sqrt{5}\times\sqrt{x+2}=2+\sqrt{5x}$. Ans. $x=\dfrac{9}{20}$.

12. Reduce $\dfrac{x-ax}{\sqrt{x}}=\dfrac{\sqrt{x}}{x}$. Ans. $x=\dfrac{1}{1-a}$.

13. Reduce $\dfrac{\sqrt{x}+28}{\sqrt{x}+4}=\dfrac{\sqrt{x}+38}{\sqrt{x}+6}$. Ans. $x=4$.

14. Reduce $\sqrt{x}+\sqrt{a+x}=\frac{2a}{\sqrt{a+x}}$. Ans. $x=\frac{1}{3}a$.

15. Reduce $x+\sqrt{a^2+x^2}=\frac{2a^2}{\sqrt{a^2+x^2}}$. Ans. $x=a\sqrt{\frac{1}{3}}$.

16. Reduce $x+a=\sqrt{a^2+x\sqrt{b^2+x^2}}$. Ans. $x=\frac{b^2-4a^2}{4a}$.

17. Reduce $\sqrt{2+x}+\sqrt{x}=\frac{4}{\sqrt{2+x}}$. Ans. $x=\frac{2}{3}$.

18. Reduce $\sqrt{x-32}=16-\sqrt{x}$. Ans. $x=81$

19. Reduce $\sqrt{4x+17}=2\sqrt{x}+1$. Ans. $x=16$.

20. Reduce $\frac{\sqrt{6x}-2}{\sqrt{6x}+2}=\frac{4\sqrt{6x}-9}{4\sqrt{6x}+6}$. Ans. $x=6$.

Reduction of Equations by Evolution.

313. In many equations, the letter which expresses the unknown quantity is involved to some power. Thus in the equation

$$x^2=16$$

we have the value of the *square* of x, but not of x itself. If the square root of both sides be extracted, we shall have

$$x=4.$$

The equality of the members is not affected by this reduction. For if two quantities or sets of quantities are equal, their roots are also equal.

If $(x+a)^n=b+h$, then $x+a=\sqrt[n]{b+h}$. Hence,

314. *When the expression containing the unknown quantity is a power, the equation is reduced by extracting the root of both sides,* a root of the same name as the power.

Ex. 1. Reduce the equation $6+x^2-8=7$

By transposition, $x^2=7+8-6=9$

By evolution, $x=\pm\sqrt{9}=\pm3$.

The signs + and − are both placed before $\sqrt{9}$, because an even root of an affirmative quantity is *ambiguous*. (Art. 272.)

2. Reduce the equation $5x^2-30=x^2+34$
Transposing, &c. $x^2=16$
By evolution, $x=\pm 4.$

3. Reduce the equation $a+\frac{x^2}{b}=h-\frac{x^2}{d}$

Clearing of fractions, &c. $x^2=\frac{bdh-abd}{b+d}$

By evolution, $x=\pm\left(\frac{bdh-abd}{b+d}\right)^{\frac{1}{2}}.$

4. Reduce the equation $a+dx^n=10-x^n$

Transposing, &c. $x^n=\frac{10-a}{d+1}$

By evolution, $x=\left(\frac{10-a}{d+1}\right)^{\frac{1}{n}}.$

315. From the preceding articles, it will be easy to see in what manner an equation is to be reduced, when the expression containing the unknown quantity is a power, and at the same time under a radical sign; that is, when it is a root of a power. Both involution and evolution will be necessary in this case.

Ex. 1. Reduce the equation $\sqrt[3]{x^2}=4.$
By involution, $x^2=4^3=64$
By evolution, $x=\pm\sqrt{64}=\pm 8.$

2. Reduce the equation $\sqrt{x^m-a}=h-d$
By involution, $x^m-a=h^2-2hd+d^2$
And $x^m=h^2-2hd+d^2+a$
By evolution, $x=\sqrt[m]{h^2-2hd+d^2+a}.$

3. Reduce the equation $(x+a)^{\frac{1}{2}}=\frac{a+b}{(x-a)^{\frac{1}{2}}}$

Multiplying by $(x-a)^{\frac{1}{2}}$ (Art. 291,) $(x^2-a^2)^{\frac{1}{2}}=a+b$
By involution, $x^2-a^2=a^2+2ab+b^2$
Transposing and uniting terms, $x^2=2a^2+2ab+b^2$
By evolution, $x=(2a^2+2ab+b^2)^{\frac{1}{2}}.$

Problems.

Prob. 1. A gentleman being asked his age, replied, "If you add to it ten years, and extract the square root of the sum, and from this root subtract 2, the remainder will be 6." What was his age?

By the conditions of the problem $\sqrt{x+10}-2=6$

By transposition, $\sqrt{x+10}=6+2=8$

By involution, $x+10=8^2=64$

And $x=64-10=54$

Proof (Art. 197,) $\sqrt{54+10}-2=6$.

Prob. 2. If to a certain number 22577 be added, and the square root of the sum be extracted, and from this 163 be subtracted, the remainder will be 237. What is the number?

Let $x=$ the number sought $b=163$

$a=22577$ $c=237$.

By the conditions proposed, $\sqrt{x+a}-b=c$

By transposition, $\sqrt{x+a}=c+b$

By involution, $x+a=(c+b)^2$

And $x=(c+b)^2-a$

Restoring the numbers, (Art. 47,) $x=(237+163)^2-22577$

That is, $x=160000-22577=137423$.

Proof $\sqrt{137423+22577}-163=237$.

316. When an equation is reduced by extracting an even root of a quantity, the solution does not determine whether the answer is positive or negative. (Art. 314.) But what is thus left ambiguous by the algebraic process, is frequently settled by the statement of the problem.

Prob. 3. A merchant gains in trade a sum, to which 320 dollars bears the same proportion as five times the sum does to 2500. What is the amount gained?

Let $x=$ the sum required.

$a=320$.

$b=2500$.

By the supposition, $a : x :: 5x : b$

Multiplying the extremes and means, $5x^2 = ab$

And $x = \left(\frac{ab}{5}\right)^{\frac{1}{2}}$

Restoring the numbers, $x = \left(\frac{320 \times 2500}{5}\right)^{\frac{1}{2}} = 400.$

Here the answer is not marked as ambiguous, because by the statement of the problem it is *gain*, and not loss. It must therefore be positive. This might be determined, in the present instance, even from the algebraic process. Whenever the root of x^2 is ambiguous, it is because we are ignorant whether the power has been produced by the multiplication of $+x$, or of $-x$, into itself. (Art. 273.) But here we have the multiplication actually performed. By turning back to the two first steps of the equation, we find that $5x^2$ was produced by multiplying $5x$ into x, that is $+5x$ into $+x$.

Prob. 4. The distance to a certain place is such, that if 96 be subtracted from the square of the number of miles, the remainder will be 48. What is the distance?

Let $x =$ the distance required.

By the supposition, $x^2 - 96 = 48$

Therefore, $x = \sqrt{144} = 12.$

Prob. 5. If three times the square of a certain number be divided by four, and if the quotient be diminished by 12, the remainder will be 180. What is the number?

By the supposition, $\frac{3x^2}{4} - 12 = 180.$

Therefore, $x = \sqrt{256} = 16.$

Prob. 6. What number is that, the fourth part of whose square being subtracted from 8, leaves a remainder equal to four? Ans. 4.

Prob. 7. What two numbers are those, whose sum is to the greater as 10 to 7; and whose sum multiplied into the less produces 270?

Let $10x =$ their sum.

Then $7x =$ the greater, and $3x =$ the less.

Therefore $x = 3$, and the numbers required are 21 and 9.

Prob. 8. What two numbers are those, whose difference is to the greater as $2:9$, and the difference of whose squares is 128? Ans. 18 and 14.

Prob. 9. It is required to divide the number 18 into two such parts, that the squares of those parts may be to each other as 25 to 16.

Let $x=$ the greater part.	Then $18-x=$ the less.
By the condition proposed,	$x^2:(18-x)^2::25:16$
Therefore,	$16x^2=25\times(18-x)^2$
By evolution,	$4x=5\times(18-x)$
And	$x=10$.

Prob. 10. It is required to divide the number 14 into two such parts, that the quotient of the greater divided by the less, may be to the quotient of the less divided by the greater, as $16:9$. Ans. The parts are 8 and 6.

Prob. 11. What two numbers are as 5 to 4, the sum of whose cubes is 5103?

Let $5x$ and $4x=$ the two numbers.

Then $x=3$, and the numbers are 15 and 12.

Prob. 12. Two travellers *A* and *B* set out to meet each other, *A* leaving the town *C*, at the same time that *B* left *D*. They travelled the direct road between *C* and *D*; and on meeting, it appeared that *A* had travelled 18 miles more than *B*, and that *A* could have gone *B*'s distance in $15\frac{3}{4}$ days, but *B* would have been 28 days in going *A*'s distance. Required the distance between *C* and *D*.

Let $x=$ the number of miles *A* travelled.

Then $x-18=$ the number *B* travelled.

$$\frac{x-18}{15\frac{3}{4}}=A\text{'s daily progress.}$$

$$\frac{x}{28}=B\text{'s daily progress.}$$

Therefore, $x:x-18::\frac{x-18}{15\frac{3}{4}}:\frac{x}{28}$.

This reduced gives $x=72$, *A*'s distance.

The whole distance, therefore, from *C* to $D=126$ miles.

Prob. 13. Find two numbers which are to each other as 8 to 5, and whose product is 360. Ans. 24 and 15.

Prob. 14. A gentleman bought two pieces of silk, which together measured 36 yards. Each of them cost as many shillings by the yard as there were yards in the piece, and their whole prices were as 4 to 1. What were the lengths of the pieces? Ans. 24 and 12 yards.

Prob. 15. Find two numbers which are to each other as 3 to 2; and the difference of whose fourth powers is to the sum of their cubes, as 26 to 7.

Ans. The numbers are 6 and 4.

Prob. 16. Several gentlemen made an excursion, each taking the same sum of money. Each had as many servants attending him as there were gentlemen; the number of dollars which each had was double the number of all the servants, and the whole sum of money taken out was 3456 dollars. How many gentlemen were there? Ans. 12.

Prob. 17. A detachment of soldiers from a regiment being ordered to march on a particular service, each company furnished four times as many men as there were companies in the whole regiment; but these being found insufficient, each company furnished three men more; when their number was found to be increased in the ratio of 17 to 16. How many companies were there in the regiment? Ans. 12.

Affected Quadratic Equations.

317. Equations are divided into classes, which are distinguished from each other by the power of the letter that expresses the unknown quantity. Those which contain only the *first* power of the unknown quantity are called equations of *one dimension,* or equations of the *first degree.* Those in which the highest power of the unknown quantity is a *square,* are called *quadratic,* or equations of the *second degree;* those in which the highest power is a *cube,* equations of the *third degree,* &c.

Thus $x=a+b$, is an equation of the *first* degree.

$x^2=c$, and $x^2+ax=d$, are *quadratic* equations, or equations of the *second* degree.

$x^3=h$, and $x^3+ax^2+bx=d$, are *cubic* equations, or equations of the *third degree.*

318. Equations are also divided into *pure* and *affected* equations. A pure equation contains only *one power* of the unknown quantity. This may be the first, second, third, or any other power. An affected equation contains *different powers* of the unknown quantity. Thus,

$$\begin{cases} x^2=d-b, & \text{is a pure quadratic equation.} \\ x^2+bx=d, & \text{an affected quadratic equation.} \end{cases}$$

$$\begin{cases} x^3=b-c, & \text{a pure cubic equation.} \\ x^3+ax^2+bx=h, & \text{an affected cubic equation.} \end{cases}$$

A pure equation is also called a *simple* equation. But this term has been applied in too vague a manner. By some writers, it is extended to pure equations of every degree; by others, it is confined to those of the first degree.

In a *pure* equation, all the terms which contain the unknown quantity may be united in one, (Art. 189,) and the equation, however complicated in other respects, may be reduced by the rules which have already been given. But in an *affected* equation, as the unknown quantity is raised to *different powers*, the terms containing these powers cannot be united. (Art. 244.) There are particular rules for the reduction of quadratic, cubic, and biquadratic equations. Of these, only the first will be considered at present.

319. *An affected quadratic equation is one which contains the unknown quantity in one term, and the square of that quantity in another term.*

The unknown quantity may be originally in *several* terms of the equation. But all these may be reduced to two, one containing the unknown quantity, and the other its square.

320. It has already been shown that a *pure* quadratic is solved by *extracting the root of both sides of the equation.* An *affected* quadratic may be solved in the same way, if the member which contains the unknown quantity is an *exact square.* Thus the equation

$$x^2+2ax+a^2=b+h,$$

may be reduced by evolution. For the first member is the square of a *binomial* quantity. (Art. 276.) And its root is $x+a$.

Therefore, $x+a=\sqrt{b+h}$, and by transposing a,

$$x=\sqrt{b+h}-a.$$

321. But it is not often the case, that a member of an affected quadratic equation is an exact square, till an additional term is supplied, for the purpose of making the required reduction. In the equation

$$x^2+2ax=b,$$

the side containing the unknown quantity is not a complete square. The two terms of which it is composed are indeed such as might belong to the square of a binomial quantity. But one term is *wanting*. We have then to inquire, in what way this may be supplied. From having *two* terms of the square of a binomial given, how shall we find the *third?*

Of the three terms, two are complete powers, and the other is twice the product of the roots of these powers; (Art. 109,) or which is the same thing, the product of one of the roots into twice the other. In the expression

$$x^2+2ax,$$

the term $2ax$ consists of the factors $2a$ and x. The latter is the unknown quantity. The other factor $2a$ may be considered the *co-efficient* of the unknown quantity; a co-efficient being another name for a factor. As x is the root of the first term x^2; the other factor $2a$ is *twice* the root of the third term, which is wanted to complete the square. Therefore *half* $2a$ is the root of the deficient term, and a^2 is the term itself. The square completed is

$$x^2+2ax+a^2,$$

where it will be seen that the last term a^2 is the square of half $2a$, and $2a$ is the co-efficient of x, the root of the first term.

In the same manner, it may be proved, that the last term of the square of any binomial quantity, is equal to the square of half the co-efficient of the root of the first term. From this principle is derived the following rule:

322. *To complete the square* in an affected quadratic equation: *take the square of half the co-efficient of the first power of the unknown quantity, and add it to both sides of the equation.*

Before completing the square, the known and unknown quantities must be brought on opposite sides of the equation by transposition; and the highest power of the unknown

quantity must have the affirmative sign, and be cleared of fractions, co-efficients, &c. See Arts. 325, 6, 7, 8.

After the square is completed, the equation is reduced, by extracting the square root of both sides, and transposing the known part of the binomial root. (Art. 320.)

The quantity which is added to one side of the equation, to complete the square, must be added to the other side also, to preserve the equality of the two members. (*Ax.* 1.)

323. It will be important for the learner to distinguish between what is *peculiar* in the reduction of quadratic equations, and what is common to this and the other kinds which have already been considered. The peculiar part, in the resolution of affected quadratics, is the *completing of the square.* The other steps are similar to those by which pure equations are reduced.

For the purpose of rendering the completing of the square familiar, there will be an advantage in beginning with examples in which the equation is already prepared for this step.

Ex. 1. Reduce the equation $x^2+6ax=b$

Completing the square, $x^2+6ax+9a^2=9a^2+b$

Extracting both sides, (Art. 320,) $x+3a=\pm\sqrt{9a^2+b}$

And $x=-3a\pm\sqrt{9a^2+b}.$

Here the co-efficient of x, in the first step, is $6a$;

The square of half this is $9a^2$, which being added to both sides completes the square. The equation is then reduced by extracting the root of each member, in the same manner as in Art. 314, excepting that the square here being that of a *binomial,* its root is found by the rule in Art. 277.

2. Reduce the equation $x^2-8bx=h$

Completing the square, $x^2-8bx+16b^2=16b^2+h$

Extracting both sides, $x-4b=\pm\sqrt{16b^2+h}$

And $x=4b\pm\sqrt{16b^2+h}.$

In this example, half the co-efficient of x is $4b$, the square of which $16b^2$ is to be added to both sides of the equation.

3. Reduce the equation $x^2+ax=b+h$

Completing the square, $x^2+ax+\frac{a^2}{4}=\frac{a^2}{4}+b+h$

By evolution, $x+\frac{a}{2}=\pm\left(\frac{a^2}{4}+b+h\right)^{\frac{1}{2}}$

And $x=-\frac{a}{2}\pm\left(\frac{a^2}{4}+b+h\right)^{\frac{1}{2}}$.

4. Reduce the equation $x^2-x=h-d$

Completing the square, $x^2-x+\frac{1}{4}=\frac{1}{4}+h-d$

And $x=\frac{1}{2}\pm(\frac{1}{4}+h-d)^{\frac{1}{2}}$.

Here the co-efficient of x is 1, the square of half which is $\frac{1}{4}$.

5. Reduce the equation $x^2+3x=d+6$

Completing the square, $x^2+3x+\frac{9}{4}=\frac{9}{4}+d+6$

And $x=-\frac{3}{2}\pm(\frac{9}{4}+d+6)^{\frac{1}{2}}$.

6. Reduce the equation $x^2-abx=ab-cd$

Completing the square, $x^2-abx+\frac{a^2b^2}{4}=\frac{a^2b^2}{4}+ab-cd$

And $x=\frac{ab}{2}\pm\left(\frac{a^2b^2}{4}+ab-cd\right)^{\frac{1}{2}}$.

7. Reduce the equation $x^2+\frac{ax}{b}=h$

Completing the square, $x^2+\frac{ax}{b}+\frac{a^2}{4b^2}=\frac{a^2}{4b^2}+h$

And $x=-\frac{a}{2b}\pm\left(\frac{a^2}{4b^2}+h\right)^{\frac{1}{2}}$.

By Art. 161, $\frac{ax}{b}=\frac{a}{b}\times x$. The co-efficient of x, therefore, is $\frac{a}{b}$. Half of this is $\frac{a}{2b}$, the square of which is $\frac{a^2}{4b^2}$.

8. Reduce the equation $x^2-\frac{x}{b}=7h$

Completing the square, $x^2-\frac{x}{b}+\frac{1}{4b^2}=\frac{1}{4b^2}+7h$

And $x=\frac{1}{2b}\pm\left(\frac{1}{4b^2}+7h\right)^{\frac{1}{2}}$.

Here the fraction $\frac{x}{b}=\frac{1}{b}\times x$. Therefore the co-efficient of x is $\frac{1}{b}$

324. In these and similar instances, the root of the third term of the completed square is easily found, because this root is the same half co-efficient from which the term has just been derived. (Art. 322.) Thus in the last example, half the co-efficient of x is $\frac{1}{2b}$, and this is the root of the third term $\frac{1}{4b^2}$.

325. When the first power of the unknown quantity is in *several terms*, these should be united in one, if they can be by the rules for reduction in addition. But if there are *literal* co-efficients, these may be considered as constituting, together, a *compound* co-efficient or factor, into which the unknown quantity is multiplied.

Thus $ax+bx+dx=(a+b+d)\times x$. (Art. 119.) The square of half this compound co-efficient is to be added to both sides of the equation.

1. Reduce the equation $x^2+3x+2x+x=d$
Uniting terms, $x^2+6x=d$
Completing the square, $x^2+6x+9=9+d$
And $x=-3\pm\sqrt{9+d}$.

2. Reduce the equation $x^2+ax+bx=h$
By Art. 119, $x^2+(a+b)\times x=h$

Therefore, $x^2+(a+b)\times x+\left(\frac{a+b}{2}\right)^2=\left(\frac{a+b}{2}\right)^2+h$

By evolution, $x+\frac{a+b}{2}=\pm\sqrt{\left(\frac{a+b}{2}\right)^2+h}$

And $x=-\frac{a+b}{2}\pm\sqrt{\left(\frac{a+b}{2}\right)^2+h}$.

3. Reduce the equation $x^2+ax-x=b$

By Art. 119, $x^2+(a-1)\times x=b$

Therefore, $x^2+(a-1)\times x+\left(\frac{a-1}{2}\right)^2=\left(\frac{a-1}{2}\right)^2+b$

And $x=-\frac{a-1}{2}\pm\sqrt{\left(\frac{a-1}{2}\right)^2+b}.$

326. After becoming familiar with the method of completing the square, in affected quadratic equations, it will be proper to attend to the steps which are *preparatory* to this. Here, however, little more is necessary, than an application of rules already given. The known and unknown quantities must be brought on opposite sides of the equation by transposition. And it will generally be expedient to make the square of the unknown quantity the first or leading term, as in the preceding examples. This indeed is not essential. But it will show, to the best advantage, the arrangement of the terms in the completed square.

1. Reduce the equation $a+5x-3b=3x-x^2$

Transposing and uniting terms, $x^2+2x=3b-a$

Completing the square, $x^2+2x+1=1+3b-a$

And $x=-1\pm\sqrt{1+3b-a}.$

2. Reduce the equation $\frac{x}{2}=\frac{36}{x+2}-4$

Clearing of fractions, &c. $x^2+10x=56$

Completing the square, $x^2+10x+25=25+56=81$

And $x=-5\pm\sqrt{81}=-5\pm9.$

327. If the *highest power* of the unknown quantity has any *co-efficient*, or *divisor*, it must, *before* the square is completed, by the rule in Art. 322, be freed from these, by multiplication or division, as in Arts. 184 and 188.

1. Reduce the equation $x^2+24a-6h=12x-5x^2$

Transposing and uniting terms, $6x^2-12x=6h-24a$

Dividing by 6, $x^2-2x=h-4a$

Completing the square, $x^2-2x+1=1+h-4a$

Extracting and transposing, $x=1\pm\sqrt{1+h-4a}.$

2. Reduce the equation $h+2x=d-\frac{bx^2}{a}$

Clearing of fractions, $bx^2+2ax=ad-ah$

Dividing by b, $x^2+\frac{2ax}{b}=\frac{ad-ah}{b}$

Therefore, $x^2+\frac{2ax}{b}+\frac{a^2}{b^2}=\frac{a^2}{b^2}+\frac{ad-ah}{b}$

And $x=-\frac{a}{b}\pm\left(\frac{a^2}{b^2}+\frac{ad-ah}{b}\right)^{\frac{1}{2}}$.

328. If the square of the unknown quantity is in *several terms*, the equation must be divided by *all* the co-efficients of this square, as in Art. 189.

1. Reduce the equation $bx^2+dx^2-4x=b-h$

Dividing by $b+d$, (Art. 124,) $x^2-\frac{4x}{b+d}=\frac{b-h}{b+d}$

Therefore, $x=\frac{2}{b+d}\pm\sqrt{\left(\frac{2}{b+d}\right)^2+\frac{b-h}{b+d}}$.

2. Reduce the equation $ax^2+x=h+3x-x^2$

Transp. and uniting terms, $ax^2+x^2-2x=h$

Dividing by $a+1$, $x^2-\frac{2x}{a+1}=\frac{h}{a+1}$

Comp. the square, $x^2-\frac{2x}{a+1}+\left(\frac{1}{a+1}\right)^2=\left(\frac{1}{a+1}\right)^2+\frac{h}{a+1}$

Extracting and transp. $x=\frac{1}{a+1}\pm\sqrt{\left(\frac{1}{a+1}\right)^2+\frac{h}{a+1}}$.

There is another method of completing the square, which, in many cases, particularly those in which the highest power of the unknown quantity has a co-efficient, is more simple in its application, than that given in Art. 322.

Let $ax^2+bx=d$.

If the equation be multiplied by $4a$, and if b^2 be added to both sides, it will become

$$4a^2x^2+4abx+b^2=4ad+b^2;$$

the first member of which is a complete power of $2ax+b$.

Hence,

329. *In a quadratic equation, the square may be completed, by multiplying the equation into four times the co-efficient of the highest power of the unknown quantity and adding to both sides, the square of the co-efficient of the lowest power.*

The advantage of this method is, that it avoids the introduction of *fractions*, in completing the square.

This will be seen, by solving an equation by both methods.

Let $ax^2+dx=h$,

Completing the square by the rule just given;

$$4a^2x^2+4adx+d^2=4ah+d^2,$$

Extracting the root, $2ax+d=\pm\sqrt{4ah+d^2}$

And $$x=\frac{-d\pm\sqrt{4ah+d^2}}{2a}.$$

Completing the square of the given equation by Arts. 322 and 327; $$x^2+\frac{dx}{a}+\frac{d^2}{4a^2}=\frac{h}{a}+\frac{d^2}{4a^2}$$

Extracting the root, $$x+\frac{d}{2a}=\pm\sqrt{\frac{h}{a}+\frac{d^2}{4a^2}}$$

And $$x=-\frac{d}{2a}\pm\sqrt{\frac{h}{a}+\frac{d^2}{4a^2}}.$$

If $a=1$, the rule will be reduced to this: "Multiply the equation by 4, and add to both sides the square of the co-efficient of x."

Let $x^2+dx=h$,

Completing the square, $4x^2+4dx+d^2=4h+d^2$

Extracting the root, $2x+d=\pm\sqrt{4h+d^2}$

And $$x=\frac{-d\pm\sqrt{4h+d^2}}{2}.$$

1. Reduce the equation $3x^2+5x=42$
   Completing the square, $36x^2+60x+25=529$
   Therefore, $x=3$.

2. Reduce the equation $x^2-15x=-54$
Completing the square, $4x^2-60x+225=9$
Therefore, $2x=15\pm3=18$ or 12.

330. In the square of a binomial, the first and last terms are always *positive.* For each is the square of one of the terms of the root. (Art. 109.) But every square is positive. (Art. 233.) If then $-x^2$ occurs in an equation, it can not, with this sign, form a part of the square of a binomial. But if *all* the signs in the equation be changed, the equality of the sides will be preserved, (Art. 182,) the term $-x^2$ will become positive, and the square may be completed.

1. Reduce the equation $-x^2+2x=d-h$
Changing all the signs, $x^2-2x=h-d$
Therefore, $x=1\pm\sqrt{1+h-d}$.
2. Reduce the equation $4x-x^2=-12$.
Ans. $x=2\pm\sqrt{16}$.

331. In a quadratic equation, the first term x^2 is the square of a single letter. But a binomial quantity may consist of terms, one or both of which are already powers.

Thus x^3+a is a binomial, and its square is

$$x^6+2ax^3+a^2,$$

where the index of x in the first term is twice as great as in the second. When the third term is deficient, the square may be completed in the same manner as that of any other binomial. For the middle term is twice the product of the roots of the two others.

So the square of x^n+a, is $x^{2n}+2ax^n+a^2$.

And the square of $x^{\frac{1}{n}}+a$, is $x^{\frac{2}{n}}+2ax^{\frac{1}{n}}+a^2$.

Therefore,

332. *Any equation which contains only two different powers or roots of the unknown quantity, the index of one of which is twice that of the other, may be resolved in the same manner as a quadratic equation, by completing the square.*

It must be observed, however, that in the binomial root, the letter expressing the unknown quantity may still have a fractional or integral index, so that a farther extraction, according to Art. 314, may be necessary.

1. Reduce the equation $x^4-x^2=b-a$
   Completing the square, $x^4-x^2+\frac{1}{4}=\frac{1}{4}+b-a$
   Extracting and transposing, $x^2=\frac{1}{2}\pm\sqrt{\frac{1}{4}+b-a}$
   Extracting again, (Art. 314,) $x=\pm\sqrt{\frac{1}{2}\pm\sqrt{(\frac{1}{4}+b-a)}}$.

2. Reduce the equation $x^{2n}-4bx^n=a$
   Answer $x=\pm\sqrt[n]{2b\pm\sqrt{(4b^2+a)}}$.

3. Reduce the equation $x+4\sqrt{x}=h-n$
   Completing the square, $x+4\sqrt{x}+4=h-n+4$
   Extracting and transposing, $\sqrt{x}=-2\pm\sqrt{h-n+4}$
   Involving, $x=(-2\pm\sqrt{h-n+4})^2$

4. Reduce the equation $x^{\frac{2}{n}}+8x^{\frac{1}{n}}=a+b$
   Completing the square, $x^{\frac{2}{n}}+8x^{\frac{1}{n}}+16=a+b+16$
   Extracting and transposing, $x^{\frac{1}{n}}=-4\pm\sqrt{a+b+16}$
   Involving, $x=(-4\pm\sqrt{a+b+16})^n$.

333. The solution of a quadratic equation, whether pure or affected, gives two results. For after the equation is reduced, it contains an ambiguous root. In a *pure* quadratic, this root is the *whole* value of the unknown quantity.

Thus the equation $x^2=64$
Becomes, when reduced, $x=\pm\sqrt{64}$.

That is, the value of x is either $+8$ or -8, for each of these is a root of 64. Here both the values of x are the same, except that they have contrary signs. This will be the case in every pure quadratic equation, because the whole of the second member is under the radical sign. The two values of the unknown quantity will be alike, except that one will be positive, and the other negative.

334. But in *affected* quadratics, a *part* only of one side of the reduced equation is under the radical sign. When

this part is added to, or subtracted from, that which is without the radical sign; the two results will differ in quantity, and will have their signs in some cases alike, and in others unlike.

1. The equation $x^2+8x=20$

Becomes when reduced, $x=-4\pm\sqrt{16+20}$

That is, $x=-4\pm6$.

Here the first value of x is, $-4+6=+2$ } one positive, and
And the second is, $-4-6=-10$ } the other negative.

2. The equation $x^2-8x=-15$

Becomes when reduced, $x=4\pm\sqrt{16-15}$

That is, $x=4\pm1$.

Here the first value of x is, $4+1=+5$ } both positive.
And the second is, $4-1=+3$ }

That these two values of x are correctly found, may be proved, by substituting first one and then the other, for x itself, in the original equation. (Art. 197.)

Thus $5^2-8\times5=25-40=-15$

And $3^2-8\times3=9-24=-15$.

335. The value of the unknown quantity, in an equation, is called, by mathematicians, a *root* of the equation. The term root has here a meaning, different from that given it in the section on radical quantities. The root of a *quantity* is a factor which multiplied into itself will produce that quantity. (Art. 252.) But a root of an *equation* is the value of the unknown quantity. It is that which substituted for the unknown quantity will answer the conditions of the equation. In a *pure* quadratic, the two meanings of the term root coincide; for when the equation is reduced, the *whole* value of the unknown quantity is under the radical sign. This root is also a root of the *equation*. But in *affected* quadratics, a *part only* of one side of the reduced equation is under the radical sign. (Art. 334.) The root indicated by this sign, therefore, is only a part of the root of the equation; a part of the value of the unknown quantity. In the last example above, it is $\sqrt{16-15}=1$; but $4+1$ is a root of the equation.

336. In the reduction of an affected quadratic equation, the value of the unknown quantity is frequently found to be *imaginary*.

Thus the equation $x^2-8x=-20$

Becomes, when reduced, $x=4\pm\sqrt{16-20}$

That is, $x=4\pm\sqrt{-4}$.

Here the root of the negative quantity -4 can not be assigned, (Art. 274,) and therefore the value of x can not be found. There will be the same impossibility, in every instance in which the negative part of the quantities under the radical sign is greater than the positive part.

337. Whenever *one* of the values of the unknown quantity, in a quadratic equation, is imaginary, the *other* is so also. For both are equally affected by the imaginary root.

Thus in the example above,

The first value of x is, $4+\sqrt{-4}$,

And the second is, $4-\sqrt{-4}$; each of which contains the imaginary quantity $\sqrt{-4}$.

338. An equation which when reduced contains an imaginary root, is often of use, to enable us to determine whether a proposed question admits of an answer, or involves an absurdity.

Suppose it is required to divide 8 into two such parts, that the product will be 20.

If x is one of the parts, the other will be $8-x$, (Art. 199.)

By the conditions proposed, $(8-x)\times x=20$

This becomes, when reduced, $x=4\pm\sqrt{-4}$.

Here the imaginary expression $\sqrt{-4}$ shows that an answer is impossible; and that there is an absurdity in supposing that 8 may be divided into two such parts, that their product shall be 20.

339. Although a quadratic equation has two solutions, yet both these may not always be applicable to the subject proposed. The quantity under the radical sign may be produced either from a positive or a negative root. But both

these roots may not, in every instance, belong to the problem to be solved. See Art. 316.

Divide the number 30 into two such parts, that their product may be equal to 8 times their difference.

If $x=$ the lesser part, then $30-x=$ the greater.

By the supposition, $x\times(30-x)=8\times(30-2x)$.

This reduced, gives $x=23\pm17=40$ or $6=$ the lesser part.

But as 40 can not be a part of 30, the problem can have but one real solution, making the lesser part 6, and the greater part 24.

Quadratic Equations containing two or more Unknown Quantities.

340. There is no general rule by which *all* quadratic equations of *two or more* unknown quantities can be resolved. When one of the equations containing two unknown quantities is a *simple* equation, the *value* of one of the unknown quantities may be found in this, and *substituted* for that quantity, in the other equation. This will reduce the latter to a quadratic with only one unknown quantity, which may be resolved by the ordinary rules. More expeditious methods may, in some cases, be adopted; particularly when *the sum or difference* of two quantities is given in one equation, and the sum or difference of their *squares* in another.

Ex. 1. Let $\left.\begin{array}{r} 2x+y=27 \\ 3xy=210 \end{array}\right\}$ And

From the 1st equation $x=\frac{1}{2}(27-y)$

Let this value be substituted for x in the second equation.

We then have $3y\times\frac{1}{2}(27-y)=210$,

Hence, $y^2-27y=-140$, an affected quadratic with only one unknown quantity, which when reduced gives $y=20$, and $x=3\frac{1}{2}$.

2. There is a certain number consisting of two digits. The left-hand digit is equal to 3 times the right-hand digit; and if twelve be subtracted from the number itself, the remainder will be equal to the square of the left-hand digit. What is the number?

Let $x=$ the left-hand digit, and $y=$ the right-hand digit.

As the *local* value of figures increases in a ten-fold ratio from right to left; the number required $=10x+y$

By the conditions of the problem, $x=3y$

And $10x+y-12=x^2$

The required number is, 93.

3. To find one of two quantities,
Whose sum is equal to h; and
The difference of whose squares is equal to d.

Let $x=$ the greater quantity; And $y=$ the less.

1. By the first condition, $x+y=h$
2. By the second, $x^2-y^2=d$
3. Transposing y^2 in the 2d equation, $x^2=d+y^2$
4. By evolution, (Art. 314,) $x=\sqrt{d+y^2}$
5. Transposing y in the first equation, $x=h-y$
6. Making the 4th and 5th equal, $\sqrt{d+y^2}=h-y$
7. Therefore, $y=\dfrac{h^2-d}{2h}$.

4. To find two numbers such, that
The product of their sum and difference shall be 5, and
The product of the sum of their squares and the difference of their squares shall be 65.

Let $x=$ the greater; And $y=$ the less.

1. By the first condition, $(x+y)\times(x-y)=5$
2. By the second, $(x^2+y^2)\times(x^2-y^2)=65$
3. Multiply the factors in the 1st, (Art. 111,) $x^2-y^2=5$
4. Dividing the 2d by the 3d, (Art. 114,) $x^2+y^2=13$
5. Adding the 3d and 4th, $2x^2=18$
6. Therefore, $x=3$, the greater number,
7. And $y=2$, the less.

In the 4th step, the first member of the second equation is divided by x^2-y^2; and the second member by 5, which is equal to x^2-y^2.

5. To find two numbers whose difference is 8, and product 240.

6. To find two numbers,
Whose difference shall be 12, and
The sum of their squares 1424.

Let $x=$ the greater; And $y=$ the less.

1. By the first condition, $x-y=12$
2. By the second, $x^2+y^2=1424$
3. Transposing y in the first, $x=y+12$
4. Squaring both sides, $x^2=y^2+24y+144$
5. Transposing y^2 in the second, $x^2=1424-y^2$
6. Making the 4th and 5th equal, $y^2+24y+144=1424-y^2$
7. Therefore, $y=-6\pm\sqrt{(676)}=-6\pm 26$
8. And $x=y+12=20+12=32.$

Properties of Quadratic Equations.

341. The several terms of an affected quadratic equation may be reduced to *three;* one containing the square of the unknown quantity, another, the quantity itself with its co-efficient, and the third, the sum of the known quantities, which may be expressed by a single letter. Every such equation, therefore, may be reduced to one of the four following forms, differing in the signs only.

$$\left.\begin{array}{ll} 1. & x^2+ax=b \\ 2. & x^2-ax=b \\ 3. & x^2+ax=-b \\ 4. & x^2-ax=-b \end{array}\right\} \text{These, when reduced, become,} \left\{\begin{array}{ll} 1. & x=-\tfrac{1}{2}a\pm\sqrt{\tfrac{1}{4}a^2+b} \\ 2. & x=\tfrac{1}{2}a\pm\sqrt{\tfrac{1}{4}a^2+b} \\ 3. & x=-\tfrac{1}{2}a\pm\sqrt{\tfrac{1}{4}a^2-b} \\ 4. & x=\tfrac{1}{2}a\pm\sqrt{\tfrac{1}{4}a^2-b} \end{array}\right.$$

In the first two of these forms, the roots are never imaginary; for the terms under the radical sign are both positive. But in the third and fourth forms, whenever b is greater than $\frac{1}{4}a^2$, the expression $\frac{1}{4}a^2-b$ is negative, and therefore its root is impossible.

In each of the first two forms, as $\sqrt{\frac{1}{4}a^2+b}$ is *greater* than $\frac{1}{2}a$, the square root of $\frac{1}{4}a^2$, one value of x is positive, the other negative. In the third form, as $\sqrt{\frac{1}{4}a^2-b}$, when not

imaginary, is *less* than $\frac{1}{2}a$, both values of x are negative; and in the fourth form, both are positive.

342. In the equation $x^2+ax=b$, or $x^2+ax-b=0$, the two values of x are $\begin{cases} -\frac{1}{2}a+\sqrt{\frac{1}{4}a^2+b} \\ -\frac{1}{2}a-\sqrt{\frac{1}{4}a^2+b} \end{cases}$

Therefore, transposing, $\begin{cases} x+\frac{1}{2}a-\sqrt{\frac{1}{4}a^2+b}=0 \\ x+\frac{1}{2}a+\sqrt{\frac{1}{4}a^2+b}=0. \end{cases}$

The first of these is the *difference*, and the other is the *sum* of the two expressions $x+\frac{1}{2}a$ and $\sqrt{\frac{1}{4}a^2+b}$.

This sum and difference *multiplied* together (Art. 111,) give

$$(x+\tfrac{1}{2}a)^2-(\tfrac{1}{4}a^2+b)=0.$$

Reducing this, we have

$$x^2+ax-b=0, \text{ the original equation.}$$

We have come to this result, by multiplying x minus the first value of x, into x minus the second value of x.

Let $v=-\frac{1}{2}a+\sqrt{\frac{1}{4}a^2+b}$, the first value of x.

And $v'=-\frac{1}{2}a-\sqrt{\frac{1}{4}a^2+b}$, the second value, or root of the equation,

We have then $x^2+ax-b=(x-v)\times(x-v')$,

And we obtain this general proposition; When all the terms of a quadratic equation are brought on one side, this member is the PRODUCT *of two binomial factors, each consisting of the unknown quantity and one of its values, with a contrary sign.*

Thus the roots of the equation $x^2+6x-55=0$ are 5 and -11.

$$\text{And } (x+11)\times(x-5)=x^2+6x-55.$$

Of consequence, if the first member of the equation be *divided* by one of these factors, the quotient will be the other factor. When, therefore, one of the values of the unknown quantity is found, the other value may be obtained by dividing the equation by the binomial factor of which the value already found is a part.

Ex. 1. What is the equation the roots of which are 3 and $-\frac{2}{3}$?

2. If one of the roots of the equation $x^2-15x=-54$ is 9, what is the other root?

343. Of the equation $x^2+ax=b$, the two roots are

$$-\tfrac{1}{2}a+\sqrt{\tfrac{1}{4}a^2+b}$$
$$-\tfrac{1}{2}a-\sqrt{\tfrac{1}{4}a^2+b}.$$

When these are *added* together, the radical parts having contrary signs, disappear; and the *sum* of the roots is $-a$, which, with the contrary sign, is the co-efficient of x, in the second term of the equation. Hence,

The SUM *of the roots* of a quadratic equation of three terms, the sign being changed, *is equal to the co-efficient of the second term of the equation.*

344. If the two roots above be *multiplied* together, the result, being the product of the sum and difference of two quantities, (Art. 111,) is $\frac{1}{4}a^2-(\frac{1}{4}a^2+b)$, that is, $-b$, which, with the contrary sign, is the second member of the equation $x^2+ax=b$. Hence,

The *second member* of a quadratic equation reduced to three terms, the signs being changed, *is equal to the product* of the two roots of the equation.

Examples of Quadratic Equations.

1. Reduce $3x^2-9x-4=80$. Ans. $x=7$, or -4.

2. Reduce $4x-\dfrac{36-x}{x}=46$. Ans. $x=12$, or $-\frac{3}{4}$.

3. Reduce $4x-\dfrac{14-x}{x+1}=14$. Ans. $x=4$, or $-\frac{7}{4}$.

4. Reduce $5x-\dfrac{3x-3}{x-3}=2x+\dfrac{3x-6}{2}$. Ans. $x=4$, or -1.

5. Reduce $\dfrac{16}{x}-\dfrac{100-9x}{4x^2}=3$. Ans. $x=4$, or $2\frac{1}{12}$.

6. Reduce $\dfrac{3x-4}{x-4}+1=10-\dfrac{x-2}{2}$. Ans. $x=12$, or 6.

7. Reduce $\frac{x+4}{3}-\frac{7-x}{x-3}=\frac{4x+7}{9}-1$. Ans. $x=21$, or 5.

8. Reduce $\frac{x^3-10x^2+1}{x^2-6x+9}=x-3$. Ans. $x=1$, or -28.

9. Reduce $\frac{6}{x+1}+\frac{2}{x}=3$. Ans. $x=2$.

10. Reduce $\frac{3x}{x+2}-\frac{x-1}{6}=x-9$. Ans. $x=10$.

11. Reduce $\frac{x}{a}+\frac{a}{x}=\frac{2}{a}$. Ans. $x=1\pm\sqrt{1-a^2}$.

12. Reduce $x^4+ax^2=b$. Ans. $x=\left(-\frac{a}{2}\pm\sqrt{b+\frac{a^2}{4}}\right)^{\frac{1}{2}}$.

13. Reduce $\frac{x^6}{2}-\frac{x^3}{4}=-\frac{1}{32}$. Ans. $x=\sqrt[3]{\frac{1}{4}}$.

14. Reduce $2x^{\frac{2}{3}}+3x^{\frac{1}{3}}=2$. Ans. $x=\frac{1}{8}$.

15. Reduce $\frac{1}{2}x-\frac{1}{3}\sqrt{x}=22\frac{1}{6}$. Ans. $x=49$.

16. Reduce $2x^4-x^2+96=99$. Ans. $x=\frac{1}{2}\sqrt{6}$.

17. Reduce $(10+x)^{\frac{1}{2}}-(10+x)^{\frac{1}{4}}=2$. Ans. $x=6$.

18. Reduce $3x^{2n}-2x^n=8$. Ans. $x=\sqrt[n]{2}$.

19. Reduce $2(1+x-x^2)-\sqrt{1+x-x^2}=-\frac{1}{9}$.
Ans. $x=\frac{1}{2}+\frac{1}{6}\sqrt{41}$.

20. Reduce $\sqrt[3]{x^3-a^3}=x-b$. Ans. $x=\frac{b}{2}\pm\sqrt{\frac{4a^3-b^3}{12b}}$.

21. Reduce $\frac{\sqrt{4x}+2}{4+\sqrt{x}}=\frac{4-\sqrt{x}}{\sqrt{x}}$. Ans. $x=4$.

22. Reduce $x^{\frac{6}{5}}+x^{\frac{3}{5}}=756$. Ans. $x=243$.

23. Reduce $\sqrt{2x+1}+2\sqrt{x}=\frac{21}{\sqrt{2x+1}}$. Ans. $x=4$.

24. Reduce $2\sqrt{x-a}+3\sqrt{2x}=\frac{7a+5x}{\sqrt{x-a}}$. Ans. $x=9a$.

25. Reduce $x+16-7\sqrt{x+16}=10-4\sqrt{x+16}$. Ans. $x=9$.

26. Reduce $\sqrt{x^5}+\sqrt{x^3}=6\sqrt{x}$.

Dividing by $\sqrt{x}$, $x^2+x=6$. Ans. $x=2$.

27. Reduce $\dfrac{4x-5}{x}-\dfrac{3x-7}{3x+7}=\dfrac{9x+23}{13x}$. Ans. $x=2$.

28. Reduce $\dfrac{3}{6x-x^2}+\dfrac{6}{x^2+2x}=\dfrac{11}{5x}$. Ans. $x=3$.

29. Reduce $(x-5)^3-3(x-5)^{\frac{3}{2}}=40$. Ans. $x=9$.

30. Reduce $x+\sqrt{x+6}=2+3\sqrt{x+6}$. Ans. $x=10$.

Problems producing Quadratic Equations.

Prob. 1. A merchant has a piece of cotton cloth, and a piece of silk. The number of yards in both is 110: and if the square of the number of yards of silk be subtracted from 80 times the number of yards of cotton, the difference will be 400. How many yards are there in each piece?

Let $x=$ the yards of silk.

Then $110-x=$ the yards of cotton.

By supposition, $400=80\times(110-x)-x^2$

Therefore, $x=-40\pm\sqrt{10000}=-40\pm100$.

The first value of x, is $-40+100=60$, the yards of silk;

And $110-x=110-60=50$, the yards of cotton.

The second value of x, is $-40-100=-140$; but as this is a negative quantity, it is not applicable to goods which a man has in his possession.

Prob. 2. The ages of two brothers are such, that their sum is 45 years, and their product 500. What is the age of each? Ans. 25 and 20 years.

Prob. 3. To find two numbers such, that their difference shall be 4, and their product 117.

Let $x=$ one number, and $x+4=$ the other.

By the conditions, $(x+4)\times x=117$

This reduced, gives, $x=-2\pm\sqrt{121}=-2\pm11$.

One of the numbers therefore is 9, and the other 13.

Prob. 4. A merchant having sold a piece of cloth which cost him 30 dollars, found that if the price for which he sold it were multiplied by his *gain*, the product would be equal to the cube of his gain. What was his gain?

Let $x=$ the gain.

Then $30+x=$ the price for which the cloth was sold

By the statement, $x^3=(30+x)\times x$

Therefore, $x=\frac{1}{2}\pm\sqrt{\frac{1}{4}+30}=\frac{1}{2}\pm\frac{11}{2}$.

The first value of x is $\frac{1}{2}+\frac{11}{2}=+6.$
The second value is $\frac{1}{2}-\frac{11}{2}=-5.$

As the last answer is *negative*, it is to be rejected as inconsistent with the nature of the problem, (Art. 339,) for *gain* must be considered *positive*.

Prob. 5. To find two numbers whose difference shall be 3, and the difference of their cubes 117.

Let $x=$ the less number.

Then $x+3=$ the greater.

By supposition, $(x+3)^3-x^3=117$

Expanding $(x+3)^3$ (Art. 232,) $9x^2+27x=117-27=90$

And $x=-\frac{3}{2}\pm\sqrt{\frac{49}{4}}=-\frac{3}{2}\pm\frac{7}{2}$.

The two numbers, therefore, are 2 and 5.

Prob. 6. To find two numbers whose difference shall be 12, and the sum of their squares 1424.

Ans. The numbers are 20 and 32.

Prob. 7. Two persons draw prizes in a lottery, the difference of which is 120 dollars, and the greater is to the less, as the less to 10. What are the prizes?

Ans. 40 and 160.

Prob. 8. What two numbers are those whose sum is 6, and the sum of their cubes 72? Ans. 2 and 4.

Prob. 9. Divide the number 56 into two such parts, that their product shall be 640.

Putting x for one of the parts, we have, $x=28\pm12=40$ or 16.

In this case, the two values of the unknown quantity are the two parts into which the given number was required to be divided.

Prob. 10. A gentleman bought a number of pieces of cloth for 675 dollars, which he sold again at 48 dollars by the piece, and gained by the bargain as much as one piece cost him. What was the number of pieces? Ans. 15.

Prob. 11. *A* and *B* started together, for a place 150 miles distant. *A*'s hourly progress was 3 miles more than *B*'s, and he arrived at his journey's end 8 hours and 20 minutes before *B*. What was the hourly progress of each?

Ans. 9 and 6 miles.

Prob. 12. The difference of two numbers is 6; and if 47 be added to twice the square of the less, it will be equal to the square of the greater. What are the numbers?

Ans. 17 and 11.

Prob. 13. *A* and *B* distributed 1200 dollars each, among a certain number of persons. *A* relieved 40 persons more than *B*, and *B* gave to each individual 5 dollars more than *A*. How many were relieved by *A* and *B*?

Ans. 120 by *A*, and 80 by *B*.

Prob. 14. Find two numbers whose sum is 10, and the sum of their squares 58. Ans. 7 and 3.

Prob. 15. Several gentlemen made a purchase in company for 175 dollars. Two of them having withdrawn, the bill was paid by the others, each furnishing 10 dollars more than would have been his equal share if the bill had been paid by the whole company. What was the number in the company at first? Ans. 7.

Prob. 16. A merchant bought several yards of linen for 60 dollars, out of which he reserved 15 yards, and sold the remainder for 54 dollars, gaining 10 cents a yard. How many yards did he buy, and at what price?

Ans. 75 yards, at 80 cents a yard.

Prob. 17. *A* and *B* set out from two towns, which were 247 miles distant, and travelled the direct road till they met. *A* went 9 miles a day; and the number of days which they travelled before meeting, was greater by 3, than the number of miles which *B* went in a day. How many miles did each travel? Ans. *A* went 117, and *B* 130 miles.

Prob. 18. A gentleman bought two pieces of cloth, the finer of which cost 4 shillings a yard more than the other. The finer piece cost £18; but the coarser one, which was 2 yards longer than the finer, cost only £16. How many yards were there in each piece, and what was the price of a yard of each?

Ans. There were 18 yards of the finer piece, and 20 of the coarser; and the prices were 20 and 16 shillings.

Prob. 19. A merchant bought 54 gallons of Madeira wine, and a certain quantity of Teneriffe. For the former, he gave half as many shillings by the gallon, as there were gallons of Teneriffe, and for the latter, 4 shillings less by the gallon. He sold the mixture at 10 shillings by the gallon, and lost £28 16s. by his bargain. Required the price of the Madeira, and the number of gallons of Teneriffe.

Ans. The Madeira cost 18 shillings a gallon, and there were 36 gallons of Teneriffe.

Prob. 20. If the square of a certain number be taken from 40, and the square root of this difference be increased by 10, and the sum be multiplied by 2, and the product divided by the number itself, the quotient will be 4. What is the number? Ans. 6.

Prob. 21. A person being asked his age, replied, If you add the square root of it to half of it, and subtract 12, the remainder will be nothing. What was his age? Ans. 16 years.

Prob. 22. Two casks of wine were purchased for 58 dollars, one of which contained 5 gallons more than the other, and the price by the gallon, was 2 dollars less than $\frac{1}{3}$ of the number of gallons in the smaller cask. Required the number of gallons in each, and the price by the gallon.

Ans. The numbers were 12 and 17, and the price by the gallon 2 dollars.

Prob. 23. In a parcel which contains 24 coins of silver and copper, each silver coin is worth as many cents as there are copper coins, and each copper coin is worth as many cents as there are silver coins; and the whole are worth 2 dollars and 16 cents. How many are there of each?

Ans. 6 of one, and 18 of the other.

Prob. 24. A person bought a certain number of oxen for 80 guineas. If he had received 4 more oxen for the same money, he would have paid one guinea less for each. What was the number of oxen? Ans. 16.

Prob. 25. There are two numbers such, that if the less be taken from three times the greater, the remainder will be 35; and if four times the greater be divided by three times the less +1, the quotient will be equal to the less. What are the numbers?

Prob. 26. What two numbers are those, whose difference, sum, and product are as the numbers 2, 3, and 5?

Ans. 10 and 2.

Prob. 27. Given And $\left\{\begin{array}{l}\frac{1}{3}(4x+2y)=6\\ 5xy\quad=50\end{array}\right\}$ To find the values of x and y.

Ans. $y=5$, or 4, $x=2$, or $2\frac{1}{2}$.

Prob. 28. Given $\left\{\begin{array}{l}x+y=a\\ x^2+y^2=b\end{array}\right\}$ To find the values of x and y.

Substitution.

345. In the reduction of Quadratic Equations, as well as in other parts of Algebra, a complicated process may be rendered much more simple, by introducing a new letter which shall be made to represent several others. This is termed *substitution*. A letter may be put for a compound quantity as well as for a single number. Thus in the equation

$$x^2-2ax=\tfrac{3}{4}+\sqrt{86-64}+h,$$

we may substitute b, for $\frac{3}{4}+\sqrt{86-64}+h$. The equation will then become $x^2-2ax=b$, and when reduced will be $x=a\pm\sqrt{a^2+b}$.

After the operation is completed, the compound quantity for which a single letter has been substituted, may be *restored*. The last equation, by restoring the value of b, will become

$$x=a\pm\sqrt{a^2+\tfrac{3}{4}+\sqrt{86-64}+h}.$$

Reduce the equation $ax-2x-d=bx-x^2-x$

Transposing, &c. $x^2+(a-b-1)\times x=d$

Substituting h for $(a-b-1)$, $x^2+hx=d$

Therefore, $x=-\dfrac{h}{2}\pm\sqrt{\dfrac{h^2}{4}+d}$

Restoring the value of h, $x=-\dfrac{a-b-1}{2}\pm\sqrt{\dfrac{(a-b-1)^2}{4}+d}$.

SECTION XI.

RATIO AND PROPORTION.

Art. **346.** The design of mathematical investigations, is to arrive at the knowledge of particular quantities, by comparing them with other quantities, either *equal* to, or *greater* or *less* than those which are the objects of inquiry. The end is most commonly attained by means of a series of *equations* and *proportions.* When we make use of equations, we determine the quantity sought, by discovering its *equality* with some other quantity or quantities already known.

We have frequent occasion, however, to compare the unknown quantity with others which are not *equal* to it, but either greater or less. Here a different mode of proceeding becomes necessary. We may inquire, either *how much* one of the quantities is greater than the other; or *how many times* the one contains the other. In finding the answer to either of these inquiries, we discover what is termed a *ratio* of the two quantities. One is called *arithmetical* and the other *geometrical* ratio. It should be observed, however, that both these terms have been adopted arbitrarily, merely for distinction's sake. Arithmetical ratio, and geometrical ratio are both of them applicable to arithmetic, and both to geometry.

As the whole of the extensive and important subject of proportion depends upon ratios, it is necessary that these should be clearly and fully understood.

347. Arithmetical ratio *is the* difference *between two quantities or sets of quantities.* The quantities themselves are called the *terms* of the ratio, that is, the terms between which the ratio exists. Thus 2 is the arithmetical ratio of 5 to 3. This is sometimes expressed, by placing two points between the quantities thus, $5..3$, which is the same as $5-3$. Indeed the term arithmetical ratio, and its notation by points, are almost needless. For the one is only a substitute for the word *difference*, and the other for the sign $-$.

348. If both the terms of an arithmetical ratio be *multiplied* or *divided* by the same quantity, the *ratio* will, in effect, be multiplied or divided by that quantity.

Thus if $a-b=r$

Then multiply both sides by h, (Ax. 3,) $ha-hb=hr$

And dividing by h, (Ax. 4,) $\frac{a}{h}-\frac{b}{h}=\frac{r}{h}$.

349. If the terms of one arithmetical ratio be added to, or subtracted from, the corresponding terms of another, the ratio of their sum or difference will be equal to the sum or difference of the two ratios.

$$\left.\begin{array}{ll}\text{If} & a-b \\ \text{And} & d-h\end{array}\right\}\text{ are the two ratios,}$$

Then $(a+d)-(b+h)=(a-b)+(d-h)$. For each $=a+d-b-h$.
And $(a-d)-(b-h)=(a-b)-(d-h)$. For each $=a-d-b+h$.

$$\left.\begin{array}{l}\text{Thus the arithmetical ratio of } 11..4 \text{ is } 7 \\ \text{And the arithmetical ratio of } 5..2 \text{ is } 3\end{array}\right\}$$

The ratio of the sum of the terms 16..6 is 10, the sum of the ratios.

The ratio of the difference of the terms 6..2 is 4, the difference of the ratios.

350. GEOMETRICAL RATIO *is that relation between quantities which is expressed by the* QUOTIENT *of the one divided by the other.**

Thus the ratio of 8 to 4, is $\frac{8}{4}$ or 2. For this is the quotient of 8 divided by 4. In other words, it shows how often 4 is contained in 8.

In the same manner, the ratio of any quantity to another may be expressed by dividing the former by the latter, or, which is the same thing, making the former the numerator of a fraction, and the latter the denominator.

Thus the ratio of a to b is $\frac{a}{b}$.

The ratio of $d+h$ to $b+c$, is $\frac{d+h}{b+c}$.

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* See Note F.

351. Geometrical ratio is also expressed by placing two points, one over the other, between the quantities compared.

Thus $a : b$ expresses the ratio of a to b; and $12 : 4$ the ratio of 12 to 4. The two quantities together are called a *couplet*, of which the first term is the *antecedent*, and the last, the *consequent*.

352. This notation by points, and the other in the form of a fraction, may be exchanged the one for the other, as convenience may require; observing to make the antecedent of the couplet, the numerator of the fraction, and the consequent the denominator.

Thus $10 : 5$ is the same as $\frac{10}{5}$ and $b : d$, the same as $\frac{b}{d}$.

353. Of these three, the antecedent, the consequent, and the ratio, any *two* being given, the other may be found.

Let $a=$ the antecedent, $c=$ the consequent, $r=$ the ratio.

By definition $r=\frac{a}{c}$; that is, the ratio is equal to the antecedent divided by the consequent.

Multiplying by c, $a=cr$, that is, the antecedent is equal to the consequent multiplied into the ratio.

Dividing by r, $c=\frac{a}{r}$, that is, the consequent is equal to the antecedent divided by the ratio.

Cor. 1. If two couplets have their antecedents equal, and their consequents equal, their ratios must be equal.*

Cor. 2. If, in two couplets, the ratios are equal, and the antecedents equal, the consequents are equal; and if the ratios are equal and the consequents equal, the antecedents are equal.†

354. If the two quantities compared are *equal*, the ratio is a unit, or a ratio of equality. The ratio of $3\times6 : 18$ is a unit, for the quotient of any quantity divided by itself is 1.

If the antecedent of a couplet is *greater* than the consequent, the ratio is greater than a unit. For if a dividend is greater than its divisor, the quotient is greater than a unit. Thus the ratio of $18 : 6$ is 3. (Art. 123, cor.) This is called a ratio of *greater inequality*.

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* Euclid 7, 5. † Euclid 9, 5.

On the other hand, if the antecedent is *less* than the consequent, the ratio is less than a unit, and is called a ratio of *less inequality*. Thus the ratio of 2 : 3, is less than a unit, because the dividend is less than the divisor.

When one ratio of inequality is compared with another, if *both* are ratios of greater, or of less inequality, they are said to subsist *in the same sense.* But if one is a ratio of greater inequality, and the other a ratio of less inequality, they are said to subsist in *a contrary sense.*

355. INVERSE *or* RECIPROCAL *ratio is the ratio of the reciprocals of two quantities.* See Art. 43.

Thus the reciprocal ratio of 6 to 3, is $\frac{1}{6}$ to $\frac{1}{3}$, that is, $\frac{1}{6}\div\frac{1}{3}$.

The direct ratio of a to b is $\frac{a}{b}$, that is, the antecedent divided by the consequent.

The reciprocal ratio is $\frac{1}{a}:\frac{1}{b}$ or $\frac{1}{a}\div\frac{1}{b}=\frac{1}{a}\times\frac{b}{1}=\frac{b}{a}$: that is the consequent b divided by the antecedent a.

Hence a reciprocal ratio is expressed by *inverting the fraction* which expresses the direct ratio; or when the notation is by points, by *inverting the order of the terms.*

Thus a is to b, inversely, as b to a.

356. COMPOUND RATIO *is the ratio of the* PRODUCTS, *of the corresponding terms of two or more simple ratios.**

Thus the ratio of	6 : 3, is 2
And the ratio of	12 : 4, is 3
The ratio compounded of these is	72 : 12=6.

Here the compound ratio is obtained by multiplying together the two antecedents, and also the two consequents, of the simple ratios.

So the ratio compounded,

Of the ratio of	$a:b$
And the ratio of	$c:d$
And the ratio of	$h:y$
Is the ratio of	$ach:bdy=\frac{ach}{bdy}$.

* See Note G.

Compound ratio is not different in its nature from any other ratio. The term is used, to denote the *origin* of the ratio, in particular cases.

Cor. The compound ratio is equal to the product of the simple ratios.

The ratio of $a : b$, is $\frac{a}{b}$

The ratio of $c : d$, is $\frac{c}{d}$

The ratio of $h : y$, is $\frac{h}{y}$

And the ratio compounded of these is $\frac{ach}{bdy}$, which is the product of the fractions expressing the simple ratios. (Art. 158.)

357. If, in a series of ratios, the consequent of each preceding couplet, is the antecedent of the following one, *the ratio of the first antecedent to the last consequent is equal to that which is compounded of all the intervening ratios.**

Thus, in the series of ratios

$$\begin{aligned} a &: b \\ b &: c \\ c &: d \\ d &: h \end{aligned}$$

the ratio of $a : h$ is equal to that which is compounded of the ratios of $a : b$, of $b : c$, of $c : d$, of $d : h$. For the compound ratio by the last article is $\frac{abcd}{bcdh} = \frac{a}{h}$ or $a : h$. (Art. 149.)

In the same manner, all the quantities which are both antecedents and consequents will *disappear* when the fractional product is reduced to its lowest terms, and will leave the compound ratio to be expressed by the first antecedent and the last consequent.

358. A particular class of compound ratios is produced, by multiplying a simple ratio into *itself*, or into another *equal* ratio. These are termed *duplicate, triplicate, quadruplicate, &c.* according to the number of multiplications.

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* This is the particular case of compound ratio which is treated of in the 5th book of Euclid. See the editions of Simson and Playfair.

A ratio compounded of *two* equal ratios, that is, the *square* of the simple ratio, is called a *duplicate* ratio.

One compounded of *three*, that is, the *cube* of the simple ratio, is called *triplicate*, &c.

In a similar manner, the ratio of the *square roots* of two quantities, is called a *subduplicate* ratio; that of the *cube roots* a *subtriplicate* ratio, &c.

Thus the simple ratio of a to b, is $a : b$

The duplicate ratio of a to b, is $a^2 : b^2$

The triplicate ratio of a to b, is $a^3 : b^3$

The subduplicate ratio of a to b, is $\sqrt{a} : \sqrt{b}$

The subtriplicate of a to b, is $\sqrt[3]{a} : \sqrt[3]{b}$, &c.

The terms *duplicate, triplicate*, &c. ought not to be confounded with *double, triple*, &c.*

The ratio of 6 to 2, is $6 : 2 = 3$

Double this ratio, that is, *twice* the ratio, is $12 : 2 = 6$

Triple the ratio, i. e. *three times* the ratio, is $18 : 2 = 9$

But the *duplicate* ratio, i.e. the *square* of the ratio, is $6^2 : 2^2 = 9$

And the *triplicate* ratio, i.e. the *cube* of the ratio, is $6^3 : 2^3 = 27$

259. That quantities may have a ratio to each other, it is necessary that they should be so far of the same nature, as that one can properly be said to be either equal to, or greater, or less than the other. A foot has a ratio to an inch, for one is twelve times as great as the other. But it can not be said that an hour is either shorter or longer than a rod; or that an acre is greater or less than a degree. Still if these quantities are expressed by *numbers*, there may be a ratio between the numbers. There is a ratio between the number of minutes in an hour, and the number of rods in a mile.

260. Having attended to the *nature* of ratios, we have next to consider in what manner they will be affected, by varying one or both of the terms between which the comparison is made. It must be kept in mind that, when a direct ratio is expressed by a fraction, the *antecedent* of the couplet is always the *numerator*, and the *consequent* the *denominator*. It will be easy, then, to derive from the properties of frac-

* See Note H.

tions, the changes produced in ratios by variations in the quantities compared. For the ratio of the two quantities is the same as the *value* of the fractions, each being the *quotient* of the numerator divided by the denominator. (Arts. 140, 350.) Now it has been shown, (Art. 142,) that multiplying the numerator of a fraction by any quantity, is multiplying the *value* by that quantity; and that dividing the numerator is dividing the value. Hence,

361. *Multiplying the antecedent of a couplet by any quantity is multiplying the ratio by that quantity; and dividing the antecedent is dividing the ratio.*

Thus the ratio of $6 : 2$, is 3
And the ratio of $24 : 2$, is 12.

Here the antecedent and the ratio, in the last couplet, are each four times as great as in the first.

The ratio of $a : b$, is $\frac{a}{b}$

And the ratio of $na : b$, is $\frac{na}{b}$.

Cor. With a given consequent, the greater the *antecedent*, the greater the *ratio;* and on the other hand, the greater the ratio, the greater the antecedent.* See Art. 142, cor.

362. *Multiplying the consequent of a couplet by any quantity, is, in effect, dividing the ratio by that quantity; and dividing the consequent is multiplying the ratio.* For multiplying the denominator of a fraction, is dividing the value; and dividing the denominator is multiplying the value. (Art. 143.)

Thus the ratio of $12 : 2$, is 6
And the ratio of $12 : 4$, is 3.

Here the consequent in the second couplet, is *twice* as great, and the ratio only *half* as great, as in the first.

The ratio of $a : b$, is $\frac{a}{b}$

And the ratio of $a : nb$, is $\frac{a}{nb}$.

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* Euclid 8 and 10. 5. The first part of the propositions.

Cor. With a given antecedent, the greater the consequent, the less the ratio; and the greater the ratio, the less the consequent.* See Art. 143, cor.

363. From the two last articles, it is evident that *multiplying the antecedent* of a couplet, by any quantity, will have the same effect on the ratio, as *dividing the consequent* by that quantity; and *dividing the antecedent*, will have the same effect as *multiplying the consequent*. See Art. 144.

Thus the ratio of	$8 : 4$, is 2
Multiplying the antecedent by 2, the ratio of	$16 : 4$, is 4
Dividing the consequent by 2, the ratio of	$8 : 2$, is 4.

Cor. Any *factor* or *divisor* may be transferred, from the antecedent of a couplet to the consequent, or from the consequent to the antecedent, without altering the ratio.

It must be observed that, when a factor is thus transferred from one term to the other, it becomes a divisor; and when a divisor is transferred, it becomes a factor.

Thus the ratio of	$3\times6 : 9=2$	the same ratio.
Transferring the factor 3,	$6 : \frac{9}{3}=2$	

$$\left.\begin{array}{ll} \text{The ratio of} & \dfrac{ma}{y} : b=\dfrac{ma}{y}\div b=\dfrac{ma}{by} \\ \text{Transferring } y, & ma : by=ma\div by=\dfrac{ma}{by} \\ \text{Transferring } m, & a : \dfrac{by}{m}=a\div\dfrac{by}{m}=\dfrac{ma}{by} \end{array}\right\}$$

364. It is farther evident, from Arts. 362 and 363, that *if the antecedent and consequent be both multiplied, or both divided, by the same quantity, the ratio will not be altered.*† See Art. 145.

Thus the ratio of	$8 : 4=2$	
Multiplying both terms by 2,	$16 : 8=2$	the same ratio.
Dividing both terms by 2,	$4 : 2=2$	

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* Euclid 8 and 10. 5. The last part of the propositions.

† Euclid 15. 5.

$$\left.\begin{array}{ll}\text{The ratio of} & a:b=\frac{a}{b}\\ \text{Multiplying both terms by } m, & ma:mb=\frac{ma}{mb}=\frac{a}{b}\\ \text{Dividing both terms by } n, & \frac{a}{n}:\frac{b}{n}=\frac{an}{bn}=\frac{a}{b}\end{array}\right\}$$

Cor. 1. The ratio of two *fractions* which have a common denominator, is the same as the ratio of their *numerators.*

Thus the ratio of $\frac{a}{n}:\frac{b}{n}$, is the same as that of $a:b$.

Cor. 2. The *direct* ratio of two fractions which have a common numerator, is the same as the reciprocal ratio of their *denominators.*

Thus the ratio of $\frac{a}{m}:\frac{a}{n}$, is the same as $\frac{1}{m}:\frac{1}{n}$, or $n:m$.

365. From the last article, it will be easy to determine the ratio of any two fractions. If each term be multiplied by the two denominators, the ratio will be assigned in integral expressions. Thus multiplying the terms of the couplet $\frac{a}{b}:\frac{c}{d}$ by bd, we have $\frac{abd}{b}:\frac{bcd}{d}$, which becomes $ad:bc$, by cancelling equal quantities from the numerators and denominators.

366. *If to or from the terms of any couplet, there be* ADDED *or* SUBTRACTED *two other quantities having the* SAME *ratio, the sums or remainders will also have the same ratio.**

$$\left.\begin{array}{ll}\text{Let the ratio of} & a:b\\ \text{Be the same as that of} & c:d\end{array}\right\}$$

Then the ratio of the *sum* of the antecedents, to the sum of the consequents, viz. of $a+c$ to $b+d$, is also the same.

That is $\frac{a+c}{b+d}=\frac{c}{d}=\frac{a}{b}$.

1. By supposition, $\frac{a}{b}=\frac{c}{d}$

2. Multiplying by b and d, $ad=bc$

* Euclid, 5 and 6. 5.

3. Adding cd to both sides, $ad+cd=bc+cd$

4. Dividing by d, $a+c=\frac{bc+cd}{d}$

5. Dividing by $b+d$, $\frac{a+c}{b+d}=\frac{c}{d}=\frac{a}{b}$.

The ratio of the *difference* of the antecedents, to the difference of the consequents, is also the same.

That is $\frac{a-c}{b-d}=\frac{c}{d}=\frac{a}{b}$.

1. By supposition, as before, $\frac{a}{b}=\frac{c}{d}$

2. Multiplying by b and d, $ad=bc$

3. Subtracting cd from both sides, $ad-cd=bc-cd$

4. Dividing by d, $a-c=\frac{bc-cd}{d}$

5. Dividing by $b-d$, $\frac{a-c}{b-d}=\frac{c}{d}=\frac{a}{b}$.

Thus the ratio of 15 : 5, is 3
And the ratio of 9 : 3, is 3

Then adding and subtracting the terms of the two couplets,

The ratio of 15+9 : 5+3, is 3
And the ratio of 15−9 : 5−3, is 3

Here the terms of only *two* couplets have been added together. But the proof may be extended to *any number* of couplets where the ratios are equal. For, by the addition of the two first, a *new* couplet is formed, to which, upon the same principle, a third may be added, a fourth, &c. Hence,

367. If, in several couplets, the ratios are equal, *the sum of all the antecedents has the same ratio to the sum of all the consequents, which any one of the antecedents has to its consequent.**

$$\text{Thus the ratio}\begin{cases}12:6=2\\10:5=2\\8:4=2\\6:3=2\end{cases}$$

Therefore the ratio of (12+10+8+6) : (6+5+4+3)=2.

* Euclid 1 and 12. 5.

Ratios of Inequality.

368. A ratio of *greater inequality* is *diminished*, by adding the *same quantity* to both the terms.

Let the given ratio be that of $a+b : a$ or $\frac{a+b}{a}$

Adding x to both terms, it becomes $a+b+x : a+x$ or $\frac{a+b+x}{a+x}$

Reducing them to a common denominator,

The first becomes $$\frac{a^2+ab+ax+bx}{a(a+x)}$$

And the latter, $$\frac{a^2+ab+ax}{a(a+x)}.$$

As the latter numerator is manifestly less than the other, the *ratio* must be less. (Art. 362, cor.)

But a ratio of *lesser inequality* is *increased*, by adding the same quantity to both terms.

Let the given ratio be that of $a-b : a$, or $\frac{a-b}{a}$

Adding x to both terms, it becomes $a-b+x : a+x$ or $\frac{a-b+x}{a+x}$

Reducing them to a common denominator,

The first becomes $$\frac{a^2-ab+ax-bx}{a(a+x)}$$

And the latter, $$\frac{a^2-ab+ax}{a(a+x)}.$$

As the latter numerator is greater than the other, the *ratio* is greater.

On the other hand, a ratio of greater inequality is *increased*, and one of lesser inequality is *diminished*, by *subtracting* from both the terms any quantity less than either of them.

369. If the same quantity be added to the two terms of an inequality, the ratio will not be changed from a greater to a lesser inequality, or the contrary; but will subsist *in the same sense*, after the addition.

If in the ratio of $c : d$, the antecedent c is greater than the consequent d, then it is evident that $c+x$ is greater

than $d+x$, (Art. 58, ax. 7.) Though the ratio, according to the preceding article, is *diminished* by the addition of x, yet it is not changed to a contrary inequality.

If in the ratio of $m:n$, m is *less* than n, it is evident that $m+x$ is less than $n+x$. Though the ratio is *increased*, by the addition of x, yet it is not changed to a contrary inequality.

If the corresponding terms of *two ratios*, subsisting in the same sense, be *added together*, the ratio will not be changed to a contrary inequality. If the antecedent a is *greater* than the consequent b, and the antecedent c is greater than the consequent d, it is manifest that $a+c$ is greater than $b+d$. And if e is *less* than f, and g less than h, $e+g$ is evidently less than $f+h$.

370. A ratio of *greater inequality*, compounded with another ratio, *increases* it.

Let the ratio of greater inequality be that of	$1+n:1$
And any given ratio, that of	$a:b$
The ratio compounded of these, (Art. 357,) is	$a+na:b$

Which is greater than that of $a:b$ (Art. 362, cor.)

But a ratio of *lesser inequality*, compounded with another ratio, *diminishes* it.

Let the ratio of lesser inequality be that of	$1-n:1$
And any given ratio that of	$a:b$
The ratio compounded of these is	$a-na:b$

Which is less than that of $a:b$.

Thus the ratio of	$6:3$, is 2
That of	$10:2$, is 5

The ratio compounded of these is $60:6=10$.

A ratio of greater inequality compounded with one of lesser inequality *may* increase the latter so much, as to convert it into one of greater inequality.

The ratio of	$2:4$, is $\frac{1}{2}$
That of	$8:2$, is 4
The compound ratio, is	$16:8=2$

On the other hand, a ratio of *lesser* inequality compounded with one of greater inequality, may diminish the latter so much, as to convert it into one of less inequality.

Examples.

1. Which is the greatest, the ratio of $11 : 9$, or that of $44 : 35$?

2. Which is the greatest, the ratio of $a+3 : \frac{1}{6}a$, or that of $2a+7 : \frac{1}{3}a$?

3. If the antecedent of a couplet be 65, and the ratio 13, what is the consequent?

4. If the consequent of a couplet be 7, and the ratio 18, what is the antecedent?

5. What is the ratio compounded of the ratios of $3 : 7$, and $2a : 5b$, and $7x+1 : 3y-2$?

6. What is the ratio compounded of $x+y : b$, and $x-y : a+b$, and $a+b : h$? Ans. $x^2-y^2 : bh$.

7. If the ratios of $5x+7 : 2x-3$, and $x+2 : \frac{1}{2}x+3$ be compounded, will they produce a ratio of greater inequality, or of lesser inequality? Ans. A ratio of greater inequality.

8. What is the ratio compounded of $x+y : a$, and $x-y : b$, and $b : \dfrac{x^2-y^2}{a}$? Ans. A ratio of equality.

9. What is the ratio compounded of $7 : 5$, and the duplicate ratio of $4 : 9$, and the triplicate ratio of $3 : 2$? Ans. $14 : 15$.

10. What is the ratio compounded of $3 : 7$, and the triplicate ratio of $x : y$, and the subduplicate ratio of $49 : 9$? Ans. $x^3 : y^3$.

Proportion.

371. An accurate and familiar acquaintance with the doctrine of ratios, is necessary to a ready understanding of the principles of *proportion*, one of the most important of all the branches of the mathematics. In considering ratios, we compare two *quantities*, for the purpose of finding either their difference, or the quotient of the one divided by the other. But in proportion, the comparison is between two *ratios*. And this comparison is limited to such ratios as are *equal*. We do not inquire how much one ratio is *greater* or *less* than another, but whether they are the *same*. Thus the numbers 12, 6, 8, 4, are said to be proportional, because the ratio of $12 : 6$ is the same as that of $8 : 4$.

372. Proportion, then, *is an equality of ratios.* It is either *arithmetical* or *geometrical.* Arithmetical proportion is an equality of arithmetical ratios, and geometrical proportion is an equality of geometrical ratios.* Thus the numbers 6, 4, 10, 8, are in *arithmetical* proportion, because the *difference* between 6 and 4 is the same as the difference between 10 and 8. And the numbers 6, 2, 12, 4, are in *geometrical* proportion, because the *quotient* of 6 divided by 2, is the same as the quotient of 12 divided by 4.

373. Care must be taken not to confound *proportion* with *ratio.* This caution is the more necessary, as in common discourse, the two terms are used indiscriminately, or rather, proportion is used for both. The expenses of one man are said to bear a greater proportion to his income, than those of another. But according to the definition which has just been given, one proportion is neither greater nor less than another. For *equality* does not admit of degrees. One *ratio* may be greater or less than another. The ratio of 12 : 2 is greater than that of 6 : 2, and less than that of 20 : 2. But these differences are not applicable to *proportion,* when the term is used in its technical sense. The loose signification which is so frequently attached to this word, may be proper enough in *familiar language:* for it is sanctioned by a general usage. But for scientific purposes, the distinction between proportion and ratio should be clearly drawn, and cautiously observed.

374. The equality between two ratios, as has been stated, is called proportion. The word is sometimes applied also to the series of terms among which this equality of ratios exists. Thus the two couplets 15 : 5 and 6 : 2 are, when taken together, called a proportion.

375. Proportion may be expressed, either by the common sign of equality, or by four points between the two couplets.

$$\text{Thus}\ \left\{\begin{array}{l} 8\cdot\cdot6=4\cdot\cdot2, \text{ or } 8\cdot\cdot6::4\cdot\cdot2 \\ a\cdot\cdot b=c\cdot\cdot d, \text{ or } a\cdot\cdot b::c\cdot\cdot d \end{array}\right\}\ \text{are arithmetical proportions.}$$

$$\text{And}\ \left\{\begin{array}{l} 12:6=8:4, \text{ or } 12:6::8:4 \\ a:b=d:h, \text{ or } a:b::d:h \end{array}\right\}\ \text{are geometrical proportions.}$$

The latter is read, 'the ratio of a to b equals the ratio of d to h;' or more concisely, 'a is to b, as d to h.'

* See Note I.

376. The first and last terms are called the *extremes*, and the other two the *means*. *Homologous* terms are either the two antecedents or the two consequents. *Analogous* terms are the antecedent and consequent of the same couplet.

377. As the ratios are equal, it is manifestly immaterial which of the two couplets is placed first.

If $a : b :: c : d$, then $c : d :: a : b$. For if $\frac{a}{b} = \frac{c}{d}$ then $\frac{c}{d} = \frac{a}{b}$.

378. The number of terms must be, at least, four. For the equality is between the ratios of *two couplets;* and each couplet must have an antecedent and a consequent. There may be a proportion, however, among three *quantities.* For one of the quantities may be *repeated,* so as to form two terms. In this case the quantity repeated is called the *middle term,* or a *mean proportional* between the two other quantities, especially if the proportion is geometrical.

Thus the numbers 8, 4, 2, are proportional. That is, $8 : 4 :: 4 : 2$. Here 4 is both the consequent in the first couplet, and the antecedent in the last. It is therefore a mean proportional between 8 and 2.

The *last* term is called a *third proportional* to the two other quantities. Thus 2 is a third proportional to 8 and 4.

379. *Inverse* or *reciprocal* proportion is an equality between a *direct* ratio, and a *reciprocal* ratio.

Thus $4 : 2 :: \frac{1}{3} : \frac{1}{6}$; that is, 4 is to 2, *reciprocally,* as 3 to 6. Sometimes also, the order of the terms in one of the couplets, is inverted, without writing them in the form of a fraction. (Art. 355.)

Thus $4 : 2 :: 3 : 6$ inversely. In this case, the *first* term is to the *second,* as the *fourth* to the *third;* that is, the first divided by the second, is equal to the fourth divided by the third.

380. When there is a series of quantities, such that the ratios of the first to the second, of the second to the third, of the third to the fourth, &c. are *all equal;* the quantities are said to be in *continued proportion.* The consequent of each preceding ratio is, then, the antecedent of the following one.—Continued proportion is also called *progression,* as will be seen in a following section.

Thus the numbers 10, 8, 6, 4, 2, are in continued *arithmetical* proportion. For $10-8=8-6=6-4=4-2$.

The numbers 64, 32, 16, 8, 4, are in continued *geometrical* proportion. For $64 : 32 :: 32 : 16 :: 16 : 8 :: 8 : 4$.

If a, b, c, d, h, &c. are in continued geometrical proportion; then $a : b :: b : c :: c : d :: d : h$, &c.

One case of continued proportion is that of *three* proportional quantities. (Art. 378.)

381. As an *arithmetical* proportion is, generally, nothing more than a very simple equation, it is scarcely necessary to give the subject a separate consideration.

The proportion $\quad a \cdot\cdot b :: c \cdot\cdot d$

Is the same as the equation $\quad a-b=c-d$

It will be proper, however, to observe that, if *four* quantities are in arithmetical proportion, ***the sum of the extremes is equal to the sum of the means.***

Thus if $a \cdot\cdot b :: h \cdot\cdot m$, then $\quad a+m=b+h$

For by supposition, $\quad a-b=h-m$

And transposing $-b$ and $-m$, $\quad a+m=b+h$

So in the proportion, $12 \cdot\cdot 10 :: 11 \cdot\cdot 9$, we have $12+9=10+11$.

Again if *three* quantities are in arithmetical proportion, ***the sum of the extremes is equal to double the mean.***

If $a \cdot\cdot b :: b \cdot\cdot c$, then $\quad a-b=b-c$

And transposing $-b$ and $-c$, $\quad a+c=2b$.

Geometrical Proportion.

382. But if four quantities are in *geometrical* proportion, ***the* PRODUCT *of the extremes is equal to the product of the means.***

If $a : b :: c : d$, $\quad ad=bc$

For by supposition, (Arts. 350, 372,) $\quad \frac{a}{b}=\frac{c}{d}$

Multiplying by bd, (Ax. 3,) $\quad \frac{abd}{b}=\frac{cbd}{d}$

Reducing the fractions, $\quad ad=bc$

Thus $12 : 8 :: 15 : 10$, therefore $12\times10=8\times15$.

Cor. Any *factor* may be transferred from one mean to the other, or from one extreme to the other, without affecting the proportion. If $a : mb :: x : y$, then $a : b :: mx : y$. For the product of the means is, in both cases, the same. And if $na : b :: x : y$, then $a : b :: x : ny$.

383. On the other hand, if the product of two quantities is equal to the product of two others, the four quantities will form a proportion, when they are so arranged, that those on one side of the equation shall constitute the means, and those on the other side, the extremes.

If $my=nh$, then $m:n::h:y$, that is, $\frac{m}{n}=\frac{h}{y}$

For by dividing $my=nh$ by ny, we have $\frac{my}{ny}=\frac{nh}{ny}$

And reducing the fractions, $\frac{m}{n}=\frac{h}{y}$.

Cor. The same must be true of *any factors* which form the two sides of an equation.

If $(a+b)\times c=(d-m)\times y$, then $a+b:d-m::y:c$.

384. If *three* quantities are proportional, the product of the extremes is equal to the square of the mean. For this mean proportional is, at the same time, the consequent of the first couplet, and the antecedent of the last. (Art. 378.) It is therefore to be multiplied *into itself*, that is, it is to be *squared*.

If $a:b::b:c$, then multiply extremes and means, $ac=b^2$.

Hence, a *mean proportional* between two quantities may be found, by *extracting the square root of their product.*

If $a:x::x:c$, then $x^2=ac$, and $x=\sqrt{ac}$. (Art. 314.)

385. It follows, from Art. 383, that in a proportion, either extreme is equal to the product of the means, divided by the other extreme; and either of the means is equal to the product of the extremes, divided by the other mean.

1. If $a:b::c:d$, then $ad=bc$

2. Dividing by d, $a=\frac{bc}{d}$

3. Dividing the first by c, $b=\frac{ad}{c}$

4. Dividing it by b, $c=\frac{ad}{b}$

5. Dividing it by a, $d=\frac{bc}{a}$; that is, the *fourth* term is equal to the *product of the second and third divided by the first.*

On this principle is founded the rule of simple proportion in arithmetic, commonly called the *Rule of Three.* Three numbers are given to find a fourth, which is obtained by multiplying together the second and third, and dividing by the first.

386. The propositions respecting the products of the means, and of the extremes, furnish a very simple and convenient criterion for determining whether any four quantities are proportional. We have only to multiply the means together, and also the extremes. If the products are equal, the quantities are proportional. If the products are not equal, the quantities are not proportional.

387. In mathematical investigations, when the relations of several quantities are given, they are frequently stated in the form of a proportion. But it is commonly necessary that this first proportion should pass through a number of transformations before it brings out distinctly the unknown quantity, or the proposition which we wish to demonstrate. It may undergo any change which will not affect the equality of the ratios; or which will leave the product of the means equal to the product of the extremes.

It is evident, in the first place, that any alteration in the *arrangement*, which will not affect the equality of these two products, will not destroy the proportion. Thus, if $a:b::c:d$, the order of these four quantities may be varied, in any way which will leave $ad=bc$. Hence,

388. If four quantities are proportional, *the order of the means, or of the extremes, or of the terms of both couplets, may be inverted without destroying the proportion.*

$$\left.\begin{array}{ll}\text{If} & a:b::c:d \\ \text{And} & 12:8::6:4\end{array}\right\}\text{then,}$$

1. *Inverting the means,**

$$\left.\begin{array}{c} a:c::b:d \\ 12:6::8:4 \end{array}\right\}\text{ that is, }\left\{\begin{array}{l}\text{The } \textit{first} \text{ is to the } \textit{third}, \\ \text{As the } \textit{second} \text{ to the } \textit{fourth}.\end{array}\right.$$

In other words, the ratio of the *antecedents* is equal to the ratio of the *consequents.*

This inversion of the means is frequently referred to by geometers, under the name of *Alternation.*†

* See Note K.

† Euclid, 16. 5.

2. *Inverting the extremes,*

$d : b :: c : a$ } that is, { The *fourth* is to the *second*,
$4 : 8 :: 6 : 12$ } { As the *third* to the *first*.

3. *Inverting the terms of each couplet,*

$b : a :: d : c$ } that is, { The *second* is to the *first*,
$8 : 12 :: 4 : 6$ } { As the *fourth* to the *third*.

This is technically called *Inversion*.

Each of these may also be varied, by changing the order of the *two couplets*. (Art. 378.)

Cor. The order of the *whole proportion* may be inverted.

If $a : b :: c : d$, then $d : c :: b : a$.

In each of these cases, it will be at once seen that, by taking the products of the means, and of the extremes, we have $ad=bc$, and $12\times4=8\times6$.

If the terms of only *one* of the couplets are inverted, the proportion becomes *reciprocal*. (Art. 379.)

If $a : b :: c : d$, then a is to b, reciprocally, as d to c.

389. A difference of arrangement is not the *only* alteration which we have occasion to produce, in the terms of a proportion. It is frequently necessary to multiply, divide, involve, &c. In all cases, the art of conducting the investigation consists in so ordering the several changes, as to maintain a constant equality, between the ratio of the two first terms, and that of the two last. As in resolving an equation, we must see that the *sides* remain equal; so in varying a proportion, the equality of the *ratios* must be preserved. And this is effected either by keeping the ratios the *same*, while the *terms* are altered; or by increasing or diminishing *one* of the ratios *as much as the other*. Most of the succeeding proofs are intended to bring this principle distinctly into view, and to make it familiar. Some of the propositions might be demonstrated, in a more simple manner, perhaps, by multiplying the extremes and means. But this would not give so clear a view of the *nature* of the several changes in the proportions.

It has been shown that, if *both* the terms of a couplet be multiplied or divided by the same quantity, the ratio will remain the same; (Art. 364,) that multiplying the *antecedent* is, in effect, multiplying the ratio, and dividing the antecedent, is dividing the ratio; (Art. 361,) and farther, that multiplying the *consequent*, is, in effect, dividing the ratio, and

dividing the consequent is multiplying the ratio. (Art. 362.) As the ratios in a proportion are equal, if they are both multiplied, or both divided, by the same quantity, they will still be equal. (Ax. 3.) One will be increased or diminished as much as the other. Hence,

390. If four quantities are proportional, *two analogous or two homologous terms may be multiplied or divided by the same quantity without destroying the proportion.*

If *analogous* terms be multiplied or divided, the ratios will not be altered. (Art. 364.) If *homologous* terms be multiplied or divided, both ratios will be equally increased or diminished. (Arts. 361, 2.)

If $a:b::c:d$, then,

1. Multiplying the two first terms,	$ma : mb :: c : d$
2. Multiplying the two last terms,	$a : b :: mc : md$
3. Multiplying the two antecedents,*	$ma : b :: mc : d$
4. Multiplying the two consequents,	$a : mb :: c : md$
5. Dividing the two first terms,	$\frac{a}{m} : \frac{b}{m} :: c : d$
6. Dividing the two last terms,	$a : b :: \frac{c}{m} : \frac{d}{m}$
7. Dividing the two antecedents,	$\frac{a}{m} : b :: \frac{c}{m} : d$
8. Dividing the two consequents,	$a : \frac{b}{m} :: c : \frac{d}{m}$.

Cor. 1. *All* the terms may be multiplied or divided by the same quantity.†

$$ma : mb :: mc : md, \quad \frac{a}{m} : \frac{b}{m} :: \frac{c}{m} : \frac{d}{m}.$$

Cor. 2. In any of the cases in this article, multiplication of the consequent may be substituted for division of the antecedent in the same couplet, and division of the consequent, for multiplication of the antecedent. (Art. 363, cor.)

$$\text{Thus for} \left\{ \begin{array}{l} ma : b :: mc : d \\ \frac{a}{m} : b :: \frac{c}{m} : d \end{array} \right\} \text{may be put} \left\{ \begin{array}{l} a : \frac{b}{m} :: mc : d \\ a : mb :: \frac{c}{m} : d \end{array} \right\} \text{or} \left\{ \begin{array}{l} ma : b :: c : \frac{d}{m} \\ \frac{a}{m} : b :: c : md \end{array} \right.$$

* Euclid, 3. 5. † Euclid, 4. 5.

391. It is often necessary not only to alter the terms of a proportion, and to vary the arrangement, but to *compare one proportion with another.* From this comparison will frequently arise a *new* proportion, which may be requisite in solving a problem, or in carrying forward a demonstration. One of the most important cases is that in which two of the terms in one of the proportions compared, are the *same* with two in the other. The similar terms may be made to *disappear*, and a new proportion may be formed of the four remaining terms. For,

392. *If two ratios are respectively equal to a third, they are equal to each other.**

This is nothing more than the 11th axiom applied to ratios.

$$\left.\begin{array}{l}\text{1. If } a:b::m:n \\ \text{And } c:d::m:n\end{array}\right\}\text{ then } a:b::c:d, \text{ or } a:c::b:d. \text{ (Art. 388.)}$$

$$\left.\begin{array}{l}\text{2. If } a:b::m:n \\ \text{And } m:n::c:d\end{array}\right\}\text{ then } a:b::c:d, \text{ or } a:c::b:d.$$

$$\left.\begin{array}{l}\textit{Cor.}\text{ If } a:b::m:n \\ \text{And } m:n>c::d\end{array}\right\}\text{ then } a:b>c:d.\dagger$$

For if the ratio of $m:n$ is greater than that of $c:d$, it is manifest that the ratio of $a:b$, which is *equal* to that of $m:n$, is also greater than that of $c:d$.

393. In these instances, the terms which are alike in the two proportions are the two *first* and the two *last.* But this arrangement is not essential. The order of the terms may be changed in various ways, without affecting the equality of the ratios.

1. The similar terms may be the two *antecedents*, or the two *consequents*, in each proportion. Thus,

$$\left.\begin{array}{l}\text{If } m:a::n:b \\ \text{And } m:c::n:d\end{array}\right\}\text{ then }\left\{\begin{array}{ll}\text{By alternation,} & m:n::a:b \\ \text{And} & m:n::c:d\end{array}\right.$$

Therefore $a:b::c:d$, or $a:c::b:d$, by the last article.

2. The *antecedents* in one of the proportions, may be the same as the *consequents* in the other.

$$\left.\begin{array}{l}\text{If } m:a::n:b \\ \text{And } c:m::d:n\end{array}\right\}\text{ then }\left\{\begin{array}{ll}\text{By inver. and altern.} & a:b::m:n \\ \text{By alternation,} & c:d::m:n\end{array}\right.$$

Therefore $a:b$, &c. as before.

* Euclid, 11. 5. † Euclid, 13. 5.

3. Two *homologous* terms, in one of the proportions, may be the same, as two *analogous* terms in the other.

$$\left.\begin{array}{ll}\text{If} & a : m :: b : n \\ \text{And} & c : d :: m : n\end{array}\right\}\text{ then }\left\{\begin{array}{ll}\text{By alternation,} & a : b :: m : n \\ \text{And} & c : d :: m : n\end{array}\right.$$

Therefore, $a : b$, &c.

All these are instances of an *equality*, between the ratios in one proportion, and those in another. In geometry, the proposition to which they belong is usually cited by the words "*ex aequo*," or "*ex aequali*."* The second case in this article is that which in its form, most obviously answers to the explanation in Euclid. But they are all upon the same principle, and are frequently referred to, without discrimination.

394. Any number of proportions may be compared, in the same manner, if the two first or the two last terms in each preceding proportion, are the same with the two first or the two last in the following one.*

$$\left.\begin{array}{ll}\text{Thus if} & a : b :: c : d \\ \text{And} & c : d :: h : l \\ \text{And} & h : l :: m : n \\ \text{And} & m : n :: x : y\end{array}\right\}\text{ then } a : b :: x : y.$$

That is, the two first terms of the first proportion have the same ratio, as the two last terms of the last proportion. For it is manifest that the ratio of *all* the couplets is the same.

And if the terms do not stand in the same order as here, yet if they can be *reduced* to this form, the same principle is applicable.

$$\left.\begin{array}{ll}\text{Thus if} & a : c :: b : d \\ \text{And} & c : h :: d : l \\ \text{And} & h : m :: l : n \\ \text{And} & m : x :: n : y\end{array}\right\}\text{ then by alternation}\left\{\begin{array}{l}a : b :: c : d \\ c : d :: h : l \\ h : l :: m : n \\ m : n :: x : y\end{array}\right.$$

Therefore $a : b :: x : y$, as before.

In all the examples in this, and the preceding articles, the two terms in one proportion which have equals in another, are neither the two *means*, nor the two *extremes*, but one of the means, and one of the extremes; and the resulting proportion is uniformly *direct*.

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* Euclid, 22. 5.

395. But if the two means, or the two extremes, in one proportion, be the same with the means, or the extremes, in another, the four remaining terms will be *reciprocally proportional.*

If $a : m :: n : b$
And $c : m :: n : d$ $\Big\}$ then $a : c :: \frac{1}{b} : \frac{1}{d}$, or $a : c :: d : b$.

For $ab=mn$
And $cd=mn$ $\Big\}$ (Art. 382.) Therefore $ab=cd$, and $a : c :: d : b$.

In this example, the two means in one proportion, are like those in the other. But the principle will be the same, if the *extremes* are alike, or if the extremes in one proportion are like the means in the other.

If $m : a :: b : n$
And $m : c :: d : n$ $\Big\}$ then $a : c :: d : b$.

Or if $a : m :: n : b$
And $m : c :: d : n$ $\Big\}$ then $a : c :: d : b$.

The proposition in geometry which applies to this case, is usually cited by the words "*ex aequo perturbate.*"*

396. Another way in which the terms of a proportion may be varied, is by *addition* or *subtraction.*

If to or from two analogous or two homologous terms of a proportion, two other quantities having the same ratio be added or subtracted, the proportion will be preserved.†

For a ratio is not altered, by adding to it, or subtracting from it, the terms of another *equal* ratio. (Art. 366.)

If $a : b :: c : d$
And $a : b :: m : n$ $\Big\}$

Then by adding to, or subtracting from a and b, the terms of the equal ratio $m : n$, we have,

$$a+m : b+n :: c : d, \quad \text{and } a-m : b-n :: c : d.$$

And by adding and subtracting m and n, to and from c and d we have,

$$a : b :: c+m : d+n, \quad \text{and } a : b :: c-m : d-n.$$

Here the addition and subtraction are to and from *analogous* terms. But by alternation, (Art. 388,) these terms will become *homologous*, and we shall have,

$$a+m : c :: b+n : d, \quad \text{and } a-m : c :: b-n : d.$$

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* Euclid, 23. 5. † Euclid, 2. 5.

Cor. 1. This addition may, evidently, be extended to *any number* of equal ratios.*

$$\text{Thus if } a : b :: \begin{cases} c : d \\ h : l \\ m : n \\ x : y \end{cases}$$

Then $a : b :: c+h+m+x : d+l+n+y$.

Cor. 2. If $\left.\begin{array}{r} a : b :: c : d \\ \text{And } m : b :: n : d \end{array}\right\}$ then $a+m : b :: c+n : d$.†

For by alternation $\left.\begin{array}{r} a : c :: b : d \\ \text{And } m : n :: b : d \end{array}\right\}$ therefore $\begin{cases} a+m : c+n :: b : d \\ \text{or } a+m : b :: c+n : d. \end{cases}$

397. From the last article it is evident that if, in any proportion, the terms be added to, or subtracted from *each other*, that is,

If two analogous or homologous terms be added to, or subtracted from the two others, the proportion will be preserved.

Thus, if $a : b :: c : d$, and $12 : 4 :: 6 : 2$, then,

1. *Adding* the two *last* terms, to the two *first*.

	$a+c : b+d :: a : b$	$12+6 : 4+2 :: 12 : 4$
and	$a+c : b+d :: c : d$	$12+6 : 4+2 :: 6 : 2$
or	$a+c : a :: b+d : b$	$12+6 : 12 :: 4+2 : 4$
and	$a+c : c :: b+d : d$	$12+6 : 6 :: 4+2 : 2$.

2. *Adding* the two *antecedents*, to the two *consequents*.

$a+b : b :: c+d : d$	$12+4 : 4 :: 6+2 : 2$
$a+b : a :: c+d : c$, &c.	$12+4 : 12 :: 6+2 : 6$, &c.

This is called *Composition*.‡

3. *Subtracting* the two *first* terms, from the two *last*.

$c-a : a :: d-b : b$

$c-a : c :: d-b : d$, &c.

4. *Subtracting* the two *last* terms from the two *first*.

$a-c : b-d :: a : b$§

$a-c : b-d :: c : d$, &c.

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* Euclid, 2. 5. Cor.

† Euclid, 24. 5.

‡ Euclid, 18. 5.

§ Euclid, 19. 5.

5. *Subtracting* the *consequents* from the *antecedents*.

$$a-b : b :: c-d : d$$

$$a : a-b :: c : c-d, \text{ \&c.}$$

The alteration expressed by the last of these forms is called *Conversion.*

6. *Subtracting* the *antecedents* from the *consequents*.

$$b-a : a :: d-c : c$$

$$b : b-a :: d : d-c, \text{ \&c.}$$

7. Adding and subtracting,

$$a+b : a-b :: c+d : c-d.$$

That is, the sum of the two first terms, is to their difference, as the sum of the two last, to their difference.

Cor. If any compound quantities, arranged as in the preceding examples, are proportional, the simple quantities of which they are compounded are proportional also.

Thus, if $a+b : b :: c+d : d$, then $a : b :: c : d$.

This is called *Division.**

398. *If the corresponding terms of two or more ranks of proportional quantities be* MULTIPLIED *together, the product will be proportional.*

This is *compounding* ratios, (Art. 356,) or compounding proportions. It should be distinguished from what is called *composition*, which is an *addition* of the terms of a ratio. (Art. 397, 2.)

$$\begin{array}{ll} \text{If} & a : b :: c : d \\ \text{And} & h : l :: m : n \\ \hline \text{Then} & ah : bl :: cm : dn \end{array} \qquad \begin{array}{l} 12 : 4 :: 6 : 2 \\ 10 : 5 :: 8 : 4 \\ \hline 120 : 20 :: 48 : 8 \end{array}$$

For from the nature of proportion, the two ratios in the first rank are equal, and also the ratios in the second rank. And multiplying the corresponding terms is multiplying the *ratios*, (Art. 361, cor.) that is, multiplying *equals by equals;* (Ax. 3,) so that the ratios will still be equal, and therefore the four products must be proportional.

The same proof is applicable to any number of proportions.

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* Euclid, 17. 5. See Note L.

$$\text{If}\ \begin{cases} a : b :: c : d \\ h : l :: m : n \\ p : q :: x : y \end{cases}$$

Then $ahp : blq :: cmx : dny$.

From this it is evident, that if the terms of a proportion be multiplied, each into *itself*, that is, if they be *raised to any power*, they will still be proportional.

If $a : b :: c : d$	$2 : 4 :: 6 : 12$
$a : b :: c : d$	$2 : 4 :: 6 : 12$
Then $a^2 : b^2 :: c^2 : d^2$	$4 : 16 :: 36 : 144$

Proportionals will also be obtained, by *reversing* this process, that is, by extracting the *roots* of the terms.

If $a : b :: c : d$, then $\sqrt{a} : \sqrt{b} :: \sqrt{c} : \sqrt{d}$.

For taking the product of extremes and means, $ad = bc$

And extracting both sides, $\sqrt{ad} = \sqrt{bc}$

That is, (Arts. 270, 383,) $\sqrt{a} : \sqrt{b} :: \sqrt{c} : \sqrt{d}$.

Hence,

399. If several quantities are proportional, *their like powers or like roots are proportional.**

If $a : b :: c : d$

Then $a^n : b^n :: c^n : d^n$, and $\sqrt[m]{a} : \sqrt[m]{b} :: \sqrt[m]{c} : \sqrt[m]{d}$.

And $\sqrt[m]{a^n} : \sqrt[m]{b^n} :: \sqrt[m]{c^n} : \sqrt[m]{d^n}$, that is, $a^{\frac{n}{m}} : b^{\frac{n}{m}} :: c^{\frac{n}{m}} : d^{\frac{n}{m}}$.

400. If the terms in one rank of proportionals be *divided* by the corresponding terms in another rank, the quotients will be proportional.

This is sometimes called the *resolution* of ratios.

If $a : b :: c : d$	$12 : 6 :: 18 : 9$
And $h : l :: m : n$	$6 : 2 :: 9 : 3$
Then $\frac{a}{h} : \frac{b}{l} :: \frac{c}{m} : \frac{d}{n}$	$\frac{12}{6} : \frac{6}{2} :: \frac{18}{9} : \frac{9}{3}$.

This is merely *reversing* the process in Art. 398, and may be demonstrated in a similar manner.

* It must not be inferred from this, that quantities have the same *ratio* as their like powers or like roots. See Art. 358.

This should be distinguished from what geometers call *division*, which is a *subtraction* of the terms of a ratio. (Art. 397, cor.)

When proportions are compounded by multiplication, it will often be the case, that the *same factor* will be found in two analogous or two homologous terms.

$$\left.\begin{array}{ll}\text{Thus if} & a : b :: c : d \\ \text{And} & m : a :: n : c\end{array}\right\}$$

$$am : ab :: cn : cd$$

Here a is in the two first terms, and c in the two last. Dividing by these, (Art. 390,) the proportion becomes

$m : b :: n : d$. Hence,

401. In compounding proportions, *equal factors* or *divisors* in two analogous or homologous terms, may be *rejected*.

$$\text{If}\left\{\begin{array}{l} a : b :: c : d \\ b : h :: d : l \\ h : m :: l : n \end{array}\right. \qquad \begin{array}{r} 12 : 4 :: 9 : 3 \\ 4 : 8 :: 3 : 6 \\ 8 : 20 :: 6 : 15 \end{array}$$

$$\text{Then } a : m :: c : n \qquad 12 : 20 :: 9 : 15$$

This rule may be applied to the cases, to which the terms "*ex aequo*" and "*ex aequo perturbate*" refer. See Arts. 393 and 395. One of the methods may serve to verify the other.

402. The changes which may be made in proportions, without disturbing the equality of the ratios, are so numerous, that they would become burdensome to the memory, if they were not reducible to a few general principles. They are mostly produced,

1. By inverting the *order* of the terms, Art. 388.
2. By *multiplying* or *dividing* by the *same quantity*, Art. 390.
3. By comparing proportions which have *like terms*, Art. 392, 3, 4, 5.
4. By *adding* or *subtracting* the terms of equal ratios, Art. 396, 7.
5. By *multiplying* or *dividing* one proportion by another, Art. 398, 400, 1.
6. By *involving* or *extracting the roots* of the terms, Art. 399.

403. When four quantities are proportional, if the *first* be greater than the *second*, the *third* will be greater than the *fourth;* if equal, equal: if less, less.

For, the ratios of the two couplets being the same, if one is a ratio of *equality*, the other is also, and therefore the antecedent in each is *equal* to its consequent; (Art. 354,) if one is a ratio of *greater inequality*, the other is also, and therefore the antecedent in each is *greater* than its consequent; and if one is a ratio of *lesser inequality*, the other is also, and therefore the antecedent in each is *less* than its consequent.

Let $a : b :: c : d$; then if $\begin{cases} a=b, & c=d \\ a>b, & c>d \\ a<b, & c<d. \end{cases}$

Cor. 1. If the *first* be greater than the *third*, the *second* will be greater than the *fourth;* if equal, equal; if less, less.*

For by alternation, $a : b :: c : d$ becomes $a : c :: b : d$, without any alteration of the quantities. Therefore, if $a=b$, $c=d$, &c. as before.

Cor. 2. If $\left.\begin{array}{r} a : m :: c : n \\ \text{And } m : b :: n : d \end{array}\right\}$ then if $a=b$, $c=d$, &c.†

For, by equality of ratios, (Art. 393. 2,) or compounding ratios, (Arts. 398, 401.)

$a : b :: c : d$. Therefore, if $a=b$, $c=d$, &c. as before.

Cor. 3. If $\left.\begin{array}{r} a : m :: n : d \\ \text{And } m : b :: c : n \end{array}\right\}$ then if $a=b$, $c=d$, &c.‡

For, by compounding ratios, (Arts. 398, 401,)

$a : b :: c : d$. Therefore, if $a=b$, $c=d$, &c.

404. If four quantities are proportional, their *reciprocals* are proportional; and v. v.

If $a : b :: c : d$, then $\frac{1}{a} : \frac{1}{b} :: \frac{1}{c} : \frac{1}{d}$.

For in each of these proportions, we have, by reduction, $ad=bc$.

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* Euclid, 14. 5. † Euclid, 20. 5. ‡ Euclid, 21. 5.

Continued Proportion.

405. When quantities are in continued proportion, *all* the ratios are *equal.* (Art. 380.) If

$$a : b :: b : c :: c : d :: d : e,$$

the ratio of $a : b$ is the same, as that of $b : c$, of $c : d$, or of $d : e$. The ratio of the *first* of these quantities to the *last*, is equal to the *product* of all the intervening ratios; (Art. 357,) that is, the ratio of $a : e$ is equal to

$$\frac{a}{b} \times \frac{b}{c} \times \frac{c}{d} \times \frac{d}{e}$$

But as the intervening ratios are all *equal*, instead of multiplying them into each other, we may multiply any one of them into *itself;* observing to make the number of factors equal to the number of intervening ratios. Thus the ratio of $a : e$, in the example just given, is equal to

$$\frac{a}{b} \times \frac{a}{b} \times \frac{a}{b} \times \frac{a}{b} = \frac{a^4}{b^4}.$$

When several quantities are in continued proportion, the number of couplets, and of course the number of ratios, is *one less* than the number of quantities. Thus the five proportional quantities a, b, c, d, e, form four couplets containing four ratios; and the ratio of $a : e$ is equal to the ratio of $a^4 : b^4$, that is, the ratio of the fourth power of the first quantity, to the fourth power of the second. Hence,

406. If *three* quantities are proportional, *the first is to the third, as the square of the first, to the square of the second;* or as the square of the second, to the square of the third. In other words, the first has to the third, a *duplicate* ratio of the first to the second. And conversely, if the first of the three quantities is to the third, as the square of the first to the square of the second, the three quantities are proportional.

If $a : b :: b : c$, then $a : c :: a^2 : b^2$. Universally,

407. If several quantities are in continued proportion, the ratio of the first to the last is equal to one of the intervening ratios raised to a power whose index is one less than the number of quantities.

If there are *four* proportionals, a, b, c, d, them $a : d :: a^3 : b^3$
If there are *five*, a, b, c, d, e; $a : e :: a^4 : b^4$, &c.

408. If several quantities are in continued proportion, they will be proportional when the order of the whole is *inverted.* This has already been proved with respect to *four* proportional quantities. (Art. 388, cor.) It may be extended to any number of quantities.

Between the numbers,	64, 32, 16, 8, 4,
The ratios are,	2, 2, 2, 2,
Between the same inverted,	4, 8, 16, 32, 64,
The ratios are,	$\frac{1}{2}$, $\frac{1}{2}$, $\frac{1}{2}$, $\frac{1}{2}$,

So if the order of any proportional quantities be inverted, the ratios in one series will be the *reciprocals* of those in the other. For by the inversion, each antecedent becomes a consequent, and *v. v.* and the ratio of a consequent to its antecedent is the reciprocal of the ratio of the antecedent to the consequent. (Art. 355.) That the reciprocals of equal quantities are themselves equal, is evident from Ax. 4.

409. HARMONICAL OR MUSICAL PROPORTION may be considered as a species of geometrical proportion. It consists in an equality of geometrical ratios; but one or more of the terms is the *difference* between two quantities.

Three or four quantities are said to be in *harmonical proportion,* when the first is to the *last,* as the difference between the *two first,* to the difference between the *two last.*

If the *three* quantities a, b, and c, are in harmonical proportion, then $a : c :: a-b : b-c$.

If the *four* quantities a, b, c, and d, are in harmonical proportion, then $a : d :: a-b : c-d$.

Thus the three numbers 12, 8, 6, are in harmonical proportion.

And the four numbers 20, 16, 12, 10, are in harmonical proportion.

410. If, of four quantities in harmonical proportion, any three be given, the other may be found. For from the proportion,

$$a : d :: a-b : c-d,$$

by taking the product of the extremes and the means, we have $ac-ad=ad-bd$.

And this equation may be reduced so as to give the value of either of the four letters.

Thus by transposing $-ad$, and dividing by a,

$$c=\frac{2ad-bd}{a}.$$

Examples, in which the principles of proportion are applied to the solution of problems.

1. Divide the number 49 into two such parts, that the greater increased by 6, may be to the less diminished by 11; as 9 to 2.

Let $x=$ the greater, and $49-x=$ the less.

By the conditions proposed, $x+6:38-x::9:2$

Adding terms, (Art. 397, 2,) $x+6:44::9:11$

Dividing the consequents, (Art. 390, 8,) $x+6:4::9:1$

Multiplying the extremes and means, $x+6=36$. And $x=30$.

2. What number is that, to which if 1, 5, and 13, be severally added, the first sum shall be to the second, as the second to the third?

Let $x=$ the number required.

By the conditions, $x+1:x+5::x+5:x+13$

Subtracting terms, (Art. 397, 6,) $x+1:4::x+5:8$

Therefore, $8x+8=4x+20$. And $x=3$.

3. Find two numbers, the greater of which shall be to the less, as their sum to 42; and as their difference to 6.

Let x and $y=$ the numbers.

By the conditions, $x:y::x+y:42$

And $x:y::x-y:6$

By equality of ratios, $x+y:42::x-y:6$

Inverting the means, $x+y:x-y::42:6$

Adding and subtracting terms, (Art. 397, 7,) $2x:2y::48:36$

Dividing terms, (Art. 390,) $x:y::4:3$

Therefore, $3x=4y$. And $x=\frac{4y}{3}$

From the second proportion, $6x=y\times(x-y)$

Substituting $\frac{4y}{3}$ for x, $y=24$. And $x=32$.

4. Divide the number 18 into two such parts, that the squares of those parts may be in the ratio of 25 to 16.

Let $x=$ the greater part, and $18-x=$ the less.

By the conditions, $x^2 : (18-x)^2 :: 25 : 16$

Extracting, (Art. 399,) $x : 18-x :: 5 : 4$

Adding terms, $x : 18 :: 5 : 9$

Dividing terms, $x : 2 :: 5 : 1$

Therefore, $x=10$.

5. Divide the number 14 into two such parts, that the quotient of the greater divided by the less, shall be to the quotient of the less divided by the greater, as 16 to 9.

Let $x=$ the greater part, and $14-x=$ the less.

By the conditions, $\frac{x}{14-x} : \frac{14-x}{x} :: 16 : 9$

Multiplying terms, $x^2 : (14-x)^2 :: 16 : 9$

Extracting, $x : 14-x :: 4 : 3$

Adding terms, $x : 14 :: 4 : 7$

Dividing terms, $x : 2 :: 4 : 1$

Therefore, $x=8$.

6. If the number 20 be divided into two parts, which are to each other in the *duplicate* ratio of 3 to 1, what number is a mean proportional between those parts?

Let $x=$ the greater part, and $20-x=$ the less.

By the conditions, $x : 20-x :: 3^2 : 1^2 :: 9 : 1$

Adding terms, $x : 20 :: 9 : 10$

Therefore, $x=18$. And $20-x=2$

A mean propor. between 18 and 2, (Art. 384,) $=\sqrt{2\times 18}=6$.

7. There are two numbers whose product is 24, and the difference of their cubes, is to the cube of their difference, as 19 to 1. What are the numbers?

Let x and y be equal to the two numbers.

$$\left.\begin{array}{ll} \text{1. By supposition,} & xy=24 \\ \text{2. And} & x^3-y^3 : (x-y)^3 :: 19 : 1 \end{array}\right\}$$

3. Or, (Art. 232,) $x^3-y^3 : x^3-3x^2y+3xy^2-y^3 :: 19 : 1$

4. Therefore, (Art. 397, 5,) $3x^2y-3xy^2 : (x-y)^3 :: 18 : 1$

5. Dividing by $x-y$, (Art. 390, 5,) $3xy : (x-y)^2 :: 18 : 1$
6. Or, as $3xy=3\times24=72$, $72 : (x-y)^2 :: 18 : 1$
7. Multiplying extremes and means, $(x-y)^2=4$
8. Extracting, $\left.\begin{array}{r} x-y=2 \\ xy=24 \end{array}\right\}$
9. By the first condition, we have

Reducing these two equations, we have $x=6$, and $y=4$.

8. It is required to prove that $a : x :: \sqrt{2a-y} : \sqrt{y}$ on supposition that $(a+x)^2 : (a-x)^2 :: x+y : x-y$.

1. Expanding, $a^2+2ax+x^2 : a^2-2ax+x^2 :: x+y : x-y$
2. Adding and subtracting terms, $2a^2+2x^2 : 4ax :: 2x : 2y$
3. Dividing terms, $a^2+x^2 : 2ax :: x : y$
4. Trans. the factor x, (Art. 382, cor.) $a^2+x^2 : 2a :: x^2 : y$
5. Inverting the means, $a^2+x^2 : x^2 :: 2a : y$
6. Subtracting terms, $a^2 : x^2 :: 2a-y : y$
7. Extracting, $a : x :: \sqrt{2a-y} : \sqrt{y}$

9. It is required to prove that $dx=cy$, if x is to y in the triplicate ratio of $a : b$, and $a : b :: \sqrt[3]{c+x} : \sqrt[3]{d+y}$.

1. Involving terms, $a^3 : b^3 :: c+x : d+y$
2. By the first supposition, $a^3 : b^3 :: x : y$
3. By equality of ratios, $c+x : d+y :: x : y$
4. Inverting the means, $c+x : x :: d+y : y$
5. Subtracting terms, $c : x :: d : y$
6. Therefore, $dx=cy$.

10. There are two numbers whose product is 135, and the difference of their squares, is to the square of their difference, as 4 to 1. What are the numbers? Ans. 15 and 9.

11. What two numbers are those, whose difference, sum, and product, are as the numbers 2, 3, and 5, respectively? Ans. 10 and 2.

12. Divide the number 24 into two such parts, that their product shall be to the sum of their squares, as 3 to 10. Ans. 18 and 6.

13. There are two numbers which are to each other as 3 to 2. If 6 be added to the greater and subtracted from the less, the sum and remainder will be to each other, as 3 to 1. What are the numbers? Ans. 24 and 16.

14. There are two numbers whose product is 320; and the difference of their cubes, is to the cube of their difference, as 61 to 1. What are the numbers? Ans. 20 and 16.

15. There are two numbers, which are to each other, in the duplicate ratio of 4 to 3; and 24 is a mean proportional between them. What are the numbers? Ans. 32 and 18.

411. A list of the articles in this section which contain the propositions in the 5th book of Euclid.*

Prop. I.	Art. 367.	Prop. XIII.	392, cor.
II.	396.	XIV.	403, cor. 1.
III.	390.	XV.	364.
IV.	390, cor. 1.	XVI.	388.
V.	366.	XVII.	397, cor.
VI.	366.	XVIII.	397, 2.
VII.	353, cor. 1.	XIX.	397, 4.
VIII.	361, cor. 362, cor.	XX.	403, cor. 2.
IX.	353, cor. 2.	XXI.	403, cor. 3.
X.	361, cor. 362, cor.	XXII.	394.
XI.	392.	XXIII.	395.
XII.	367.	XXIV.	396, cor. 2.

SECTION XII.

VARIATION OR GENERAL PROPORTION.

ART. **412.** THE quantities which constitute the terms of a proportion are, frequently, so related to each other, that, if one of them be either increased or diminished, another depending on it will also be increased or diminished, in such a manner, that the proportion will still be preserved. If the value of 50 yards of cloth is 100 dollars, and the quantity be reduced to 40 yards; the value will, of course, be reduced to 80 dollars; if the quantity be reduced to 30 yards, the value will be reduced to 60 dollars, &c.

* See Note M.

$$\begin{array}{lcccc} & yd. & yd. & dol. & dol. \\ \text{That is,} & 50 : 40 & :: & 100 : 80 \\ & 50 : 30 & :: & 100 : 60 \\ & 50 : 20 & :: & 100 : 40, \text{ \&c.} \end{array}$$

As the consequent of the *first* couplet is varied, the consequent of the *second* is varied, in such a manner, that the proportion is constantly preserved.

If the two antecedents are A and B; and if a represents a quantity of the *same kind* with A, but either greater or less; and b, a quantity of the same kind with B, but as many times greater or less, as a is greater or less than A; then

$$A : a :: B : b;$$

that is, if A by varying becomes a, then B becomes b. This is expressed more concisely, by saying that *A varies as B*, or *A is as B.* Thus the *wages* of a laboring man vary as the *time* of his service. We say that the interest of money which is loaned for a given time, is *proportioned* to the principal. But a proportion contains *four terms.* Here are only two, the *interest* and the *principal.* This then is an *abridged statement,* in which two terms are mentioned instead of four. The proportion in form would be:

As any given principal, is to any other principal;

So is the interest of the former, to the interest of the latter.

413. In many mathematical and philosophical investigations, we have occasion to determine the general relations of certain classes of quantities to each other, without limiting the inquiry to any particular values of those quantities. In such cases, it is frequently sufficient to mention only two of the terms of a proportion. It must be kept in mind, however, that four are always *implied.* When it is said, for instance, that the weight of water is proportioned to its bulk, we are to understand,

That *one* gallon, is to any *number* of gallons;

As the *weight* of one gallon, is to the weight of the given number of gallons.

414. The character $\propto$ is used to express the proportion of variable quantities.

Thus $A \propto B$ signifies that A *varies* as B, that is, that

$$A : a :: B : b.$$

The expression $A \propto B$ may be called a *general proportion.*

415. One quantity is said to vary *directly* as another, when the one increases as the other increases, or is diminished as the other is diminished, so that

$$A \propto B, \text{ that is, } A : a :: B : b.$$

The interest on a loan is increased or diminished, in proportion to the principal. If the principal is doubled, the interest is doubled; if the principal is trebled, the interest is trebled, &c.

416. One quantity is said to vary *inversely* or *reciprocally* as another, when the one is proportioned to the reciprocal of the other; that is, when the one is diminished, as the other is increased, so that

$$A \propto \frac{1}{B}, \text{ that is, } A : a :: \frac{1}{B} : \frac{1}{b}, \text{ or } A : a :: b : B.$$

In this case, if A is greater than a, B is less than b. (Art. 403.) The time required for a man to raise a given sum, by his labor, is inversely as his wages. The higher his wages, the less the time.

417. One quantity is said to vary as *two others jointly*, when the one is increased or diminished, as the *product* of the other two, so that

$$A \propto BC, \text{ that is, } A : a :: BC : bc.$$

The interest of money varies as the product of the principal and time. If the time be doubled, and the principal doubled, the interest will be four times as great.

418. One quantity is said to vary *directly* as a *second*, and *inversely* as a *third*, when the first is always proportioned to the second divided by the third, so that

$$A \propto \frac{B}{C}, \text{ that is, } A : a :: \frac{B}{C} : \frac{b}{c}.$$

419. To understand the methods by which the statements of the relations of variable quantities are changed from one form to another, little more is necessary, than to make an *application* of the principles of common proportion; bearing constantly in mind, that a general proportion is only an abridged expression, in which two terms are mentioned instead of four. When the deficient terms are supplied, the reason of the several operations will, in most cases, be apparent.

420. It is evident, in the first place, that the *order of the terms* in a general proportion may be *inverted.* (Art. 377.)

If $A : a :: B : b$, that is, if $A \propto B$,

Then $B : b :: A : a$, that is, $B \propto A$.

421. If one or both of the terms in a general proportion, be *multiplied* or *divided* by a constant quantity, the proportion will be preserved.

For multiplying or dividing one or both of the terms is the same, as multiplying or dividing *analogous* terms in the proportion expressed at length. (Art. 390, and cor. 1.)

If $A : a :: B : b$, that is, if $A \propto B$,

Then $mA : ma :: B : b$, that is, $mA \propto B$,

And $mA : ma :: mB : mb$, that is, $mA \propto mB$, &c.

422. If *both* the terms be multiplied or divided even by a *variable* quantity, the proportion will be preserved. For this is equivalent to multiplying the two *antecedents* by one quantity, and the two *consequents* by another. (Art. 390.)

If $A : a :: B : b$, that is, if $A \propto B$,

Then $MA : ma :: MB : mb$, that is, $MA \propto MB$, &c.

Cor. 1. If one quantity varies as another, the *quotient* of the one divided by the other is *constant.* In other words, if the numerator of a fraction varies as the denominator, the *value* remains the same.

If $A : a :: B : b$, that is, if $A \propto B$,

Then $\frac{A}{B} : \frac{a}{b} :: \frac{B}{B} : \frac{b}{b} :: 1 : 1.$ (Art. 123.)

Here the third and fourth terms are equal, because each is equal to 1. Of course the two first terms are equal; (Art. 403,) so that if A be increased or diminished as many times as B, the *quotient* will be invariably the same.

Cor. 2. If the *product* of two quantities is *constant,* one varies *reciprocally* as the other.

If $AB : ab :: 1 : 1$, then $\frac{AB}{B} : \frac{ab}{b} :: \frac{1}{B} : \frac{1}{b}$, or $A : a :: \frac{1}{B} : \frac{1}{b}$.

Cor. 3. Any *factor* in one term of a general proportion, may be *transferred,* so as to become a *divisor* in the other; and *v. v.*

If $A \propto BC$, then dividing by B, $\frac{A}{B} \propto C$.

If $A \propto \frac{1}{CD}$, then multiplying by C, $AC \propto \frac{1}{D}$.

423. If two quantities vary respectively as a third, then one of the two varies as the other. (Art. 392.)

$$\left.\begin{array}{ll}\text{If} & A:a::B:b \\ \text{And} & C:c::B:b\end{array}\right\} \text{that is, if } \left\{\begin{array}{l}A \propto B \\ C \propto B;\end{array}\right.$$

$$\text{Then } A:a::C:c, \text{ that is, } A \propto C.$$

424. If two quantities vary respectively, as a third, their *sum* or *difference* will vary in the same manner. (Art. 396.)

$$\left.\begin{array}{ll}\text{If} & A:a::B:b \\ \text{And} & C:c::B:b\end{array}\right\} \text{that is, if } \left\{\begin{array}{l}A \propto B \\ C \propto B;\end{array}\right.$$

Then $A+C:a+c::B:b$, that is, $A+C \propto B$,

And $A-C:a-c::B:b$, that is, $A-C \propto B$.

Cor. The addition here may be extended to *any number* of quantities all varying alike. (Art. 396, cor. 1.)

If $A \propto B$, and $C \propto B$, and $D \propto B$, and $E \propto B$, then

$$(A+C+D+E) \propto B$$

425. If the *square of the sum* of two quantities, varies as the *square of their difference:* then the *sum of their squares* varies as their *product.*

If $(A+B)^2 \propto (A-B)^2$; then $A^2+B^2 \propto AB$.

For by the supposition,

$$(A+B)^2 : (A-B)^2 :: (a+b)^2 : (a-b)^2.$$

Expanding, adding, and subtracting terms. (Arts. 232, and 397, 7.)

$$2A^2+2B^2 : 4AB :: 2a^2+2b^2 : 4ab.$$

Or, (Art. 390.)

$A^2+B^2 : AB :: a^2+b^2 : ab$, that is, $A^2+B^2 \propto AB$.

426. The terms of one general proportion may be multiplied or divided by the corresponding terms of another. (Art. 398.)

$$\left.\begin{array}{ll}\text{If} & A:a::B:b \\ \text{And} & C:c::D:d\end{array}\right\} \text{that is, if } \left\{\begin{array}{l}A \propto B \\ C \propto D;\end{array}\right.$$

Then $AC:ac::BD:bd$, that is, $AC \propto BD$.

Cor. If two quantities vary respectively as a third, the *product* of the two will vary as the *square* of the other.

$$\left.\begin{array}{ll}\text{If} & A \backsim B \\ \text{And} & C \backsim B\end{array}\right\} \text{then } AC \backsim B^2.$$

427. If any quantity vary as another, any *power* or *root* of the former will vary, as a like power, or root of the latter. (Art. 399.)

If	$A : a :: B : b$,	that is, if $A \backsim B$,
Then	$A^n : a^n :: B^n : b^n$,	that is, $A^n \backsim B^n$,
And	$A^{\frac{1}{n}} : a^{\frac{1}{n}} :: B^{\frac{1}{n}} : b^{\frac{1}{n}}$,	that is, $A^{\frac{1}{n}} \backsim B^{\frac{1}{n}}$.

428. In compounding general proportions, equal *factors* or *divisors*, in the two terms, may be rejected. (Art. 401.)

$$\left.\begin{array}{ll}\text{If} & A : a :: B : b \\ \text{And} & B : b :: C : c \\ \text{And} & C : c :: D : d\end{array}\right\} \text{that is, if } \left\{\begin{array}{l}A \backsim B \\ B \backsim C \\ C \backsim D\end{array}\right.$$

$$\text{Then } A : a :: D : d, \text{ that is, } A \backsim D.$$

Cor. If one quantity varies as a second, the second, as a third, the third, as a fourth, &c. then the *first* varies as the last.

If $A \backsim B \backsim C \backsim D$, then $A \backsim D$.

If $A \backsim B \backsim \frac{1}{C}$, then $A \backsim \frac{1}{C}$; that is, if the first varies *directly* as the second, and the second varies *reciprocally* as the third; the first varies reciprocally as the third.

429. If any quantity vary as the *product* of two others, and if one of the latter be considered *constant*, the first will vary as the other.

If $W \backsim LB$, and if B be constant, then $W \backsim L$.

Here it must be observed that there are *two conditions;* First, that W varies as the *product* of the two other quantities; Secondly, that one of these quantities B is *constant.*

Then, by the conditions, $W : w :: LB : lB$; B being the same in both terms.

Divid. by the constant quantity B, $W : w :: L : l$, that is, $W \backsim L$.

And if L be considered constant, $W \backsim B$.

Thus the weight of a board, of uniform thickness and density, varies as its length and breadth. If the *length* is given, the weight varies as the breadth. And if the *breadth* is given, the weight varies as the length.

Cor. The same principle may be extended to any number of quantities. The weight of a stick of timber, of given density, depends on the length, breadth, and thickness. If the length is given, the weight varies as the breadth and thickness. If the length and breadth are given, the weight varies as the thickness, &c.

If $\qquad W \propto LBT,$
Then making L constant, $\qquad W \propto BT,$
And making L and B constant, $\qquad W \propto T.$

430. On the other hand, if one quantity depends on two others; so that when the second is given, the first varies as the third, and when the third is given, the first varies as the second; then the first varies as the *product* of the other two.

If the weight of a board varies as the length, when the breadth is given, and as the breadth when the length is given: then if the length and breadth *both* vary, the weight varies as their product.

$$\left.\begin{array}{ll}\text{If} & W \propto L, \text{ when } B \text{ is constant,}\\ \text{And} & W \propto B, \text{ when } L \text{ is constant,}\end{array}\right\} \text{then } W \propto BL.$$

In demonstrating this, we have to consider, *two variable values* of W; one, when L *only* varies, and the other, when L and B *both* vary.

Let $w'=$ the first of these variable values,

And $w=$ the other;

So that W will be changed to w', by the varying of L;

And w' will be farther changed to w, by the varying of B.

Then by the supposition, $W : w' :: L : l$, when B is constant.
And $\qquad w' : w :: B : b$, when B varies.

Mult. correspond. terms, $Ww' : ww' :: BL : bl.$ (Art. 398.)
Dividing by w', (Art. 390,) $W : w :: BL : bl$, i. e. $W \propto BL.$

The proof may be extended to any number of quantities.

The weight of a piece of timber, depends on its length, breadth, thickness and density. If any three of these are given, the weight varies as the other.

This case must not be confounded with that in Art. 423, cor. In that, B is supposed to vary as A and as C, *at the same time.* In this, B varies as A, only when C is constant, and as C, only when A is constant. It can not therefore vary as A and as C separately, at the same time.

If one quantity varies as another, the former is equal to the product of the latter into some *constant* quantity.

If $A : B :: a : b$; then, whatever be the value of a, its ratio to b must be constant, viz. that of $A : B$. Let this ratio be that of $m : 1$.

Then $A : B :: a : b :: m : 1$. Therefore $A = mB$; And $a = mb$.

Hence, if the ratio between the two quantities be found for any given value, it will be known for any other period of their increase or decrease. If the interest of 100 dollars be to the principal as 1 : 20; the interest of 1000 or 10,000 will have the same ratio to the principal.

431. Many writers, in expressing a general proportion, do not use the term *vary*, or the character which has here been put for it. Instead of $A \propto B$, they say simply that A *is as* B. It may be proper to observe, also, that the word *given* is frequently used to distinguish *constant* quantities, from those which are variable; as well as to distinguish *known* quantities from those which are unknown.

SECTION XIII.

ARITHMETICAL AND GEOMETRICAL PROGRESSION.

ART. **432.** QUANTITIES which decrease by a common difference, as the numbers 10. 8, 6, 4, 2, are in continued arithmetical proportion. (Art. 380.) Such a series is also called a *progression*, which is only another name for continued proportion.

It is evident that the proportion will not be destroyed, if the order of the quantities be *inverted*. Thus the numbers 2, 4, 6, 8, 10, are in arithmetical proportion.

Quantities, then, *are in arithmetical progression, when they increase or decrease by a common difference.*

When they *increase*, they form what is called an *ascending* series, as 3, 5, 7, 9, 11, &c.

When they *decrease*, they form a *descending* series, as 11, 9, 7, 5, &c.

The natural numbers, 1, 2, 3, 4, 5, 6, &c. are in arithmetical progression ascending.

433. From the definition it is evident that, in an *ascending* series, each succeeding term is found, by *adding the common difference* to the preceding term.

If the first term is 3, and the common difference 2;
The series is 3, 5, 7, 9, 11, 13, &c.
If the first term is a, and the common difference d;
Then $a+d$ is the second term, $a+2d+d=a+3d$, the fourth,
$a+d+d=a+2d$ the 3d, $a+3d+d=a+4d$, the 5th, &c.

And the series is $\overset{1}{a}$, $\overset{2}{a+d}$, $\overset{3}{a+2d}$, $\overset{4}{a+3d}$, $\overset{5}{a+4d}$, &c.

If the first term and the common difference are the *same*, the series becomes more simple. Thus if a is the first term, and the common difference, and n the number of terms,

Then $a+a=2a$ is the second term,
$2a+a=3a$ the third term, &c.

And the series is a, $2a$, $3a$, $4a$, ... na.

434. In a *descending* series, each succeeding term is found, by *subtracting the common difference* from the preceding term.

If a is the first term, and d the common difference, the series is $\overset{1}{a}$, $\overset{2}{a-d}$, $\overset{3}{a-2d}$, $\overset{4}{a-3d}$, $\overset{5}{a-4d}$, &c.

Or the common difference in this case may be considered as $-d$, a negative quantity, by the addition of which to any preceding term, we obtain the following term.

In this manner, we may obtain any term, by continued addition or subtraction. But in a long series, this process would become tedious. There is a method much more expeditious. By attending to the series

$$\overset{1}{a},\ \overset{2}{a+d},\ \overset{3}{a+2d},\ \overset{4}{a+3d},\ \overset{5}{a+4d},\ \&c.$$

it will be seen, that the number of times d is added to a is *one less* than the number of the term.

The *second* term is $a+d$, i. e. a added to *once* d;
The *third* is $a+2d$, a added to *twice* d;
The *fourth* is $a+3d$, a added to *thrice* d, &c.

So if the series be continued,
The 50th term will be $a+49d$,
The 100th term $a+99d$.

If the series be *descending*, the 100th term will be $a-99d$.

In the *last* term, the number of times d is added to a, is *one less* than the number of all the terms. If then

$a=$ the first term, $z=$ the last, $n=$ the number of terms, we shall have, in all cases, $z=a+(n-1)\times d$; that is,

435. In an arithmetical progression, *the last term is equal to the first, + the product of the common difference into the number of terms less one.*

Any other term may be found in the same way. For the series may be made to stop at any term, and that may be considered, for the time, as the last.

Thus the mth term $=a+(m-1)\times d$.

If the first term and the common difference are the *same*,

$z=a+(n-1)a=a+na-a$, that is, $z=na$.

In an *ascending series*, the first term is, evidently, the least, and the last, the greatest. But in a descending series, the first term is the greatest, and the last, the least.

436. The equation $z=a+(n-1)d$, not only shows the value of the last term, but, by a few simple reductions, will enable us to find other parts of the series. It contains four different quantities,

a, the *first* term,	n, the *number* of terms, and
z, the *last* term,	d, the *common difference.*

If any three of these be given, the other may be found.

1. By the equation already found,

$$z=a+(n-1)d= \textit{ the last term.}$$

2. Transposing $(n-1)d$, (Art. 178,)

$$z-(n-1)d=a= \textit{ the first term.}$$

3. Transposing a in the 1st, and dividing by $n-1$,

$$\frac{z-a}{n-1}=d= \textit{ the common difference.}$$

4. Transp. a in the 1st, dividing by d, and transp. -1,

$$\frac{z-a}{d}+1=n= \textit{ the number of terms.}$$

By the third equation, may be found any number of *arithmetical means*, between two given numbers. For the *whole* number of terms consists of the *two extremes*, and all the

intermediate terms. If then $m=$ the number of means, $m+2=n$, the whole number of terms. Substituting $m+2$ for n, in the third equation, we have

$$\frac{z-a}{m+1}=d, \text{ the common difference.}$$

Prob. 1. If the first term of an increasing progression is 7, the common difference 3, and the number of terms 9, what is the last term? Ans. $z=a+(n-1)d=7+(9-1)\times3=31$.

And the series is 7, 10, 13, 16, 19, 22, 25, 28, 31.

Prob. 2. If the last term of an increasing progression is 60, the number of terms 12, and the common difference 5, what is the first term? Ans. $a=z-(n-1)d=60-(12-1)\times5=5$.

Prob. 3. Find 6 arithmetical means, between 1 and 43.
Ans. The common difference is 6.

And the series, 1, 7, 13, 19, 25, 31, 37, 43.

437. There is another inquiry to be made concerning a series in arithmetical progression. It is often necessary to find the *sum of all the terms*. This is called the *summation* of the series. The most obvious mode of obtaining the amount of the terms, is to add them together. But the nature of progression will furnish us with a method more expeditious.

It is manifest that the sum of the terms will be the same, in whatever *order* they are written. The sum of the ascending series, 3, 5, 7, 9, 11, is the same, as that of the descending series, 11, 9, 7, 5, 3. The sum of *both* the series, is, therefore, *twice* as great, as the sum of the terms in one of them. There is an easy method of finding this *double sum*, and of course, the sum itself which is the object of inquiry. Let a given series be written, both in the direct, and in the inverted order, and then add the corresponding terms together.

Take, for instance, the series	3,	5,	7,	9,	11
And the same inverted,	11,	9,	7,	5,	3
The sums of the terms will be	14,	14,	14,	14,	14

Take also the series	a,	$a+d$,	$a+2d$,	$a+3d$,	$a+4d$
Take the same inverted,	$a+4d$,	$a+3d$,	$a+2d$,	$a+d$,	a
The sums will be	$2a+4d$,	$2a+4d$,	$2a+4d$,	$2a+4d$,	$2a+4d$

Here we discover the important property, that,

438. In an arithmetical progression, *the sum of the extremes is equal to the sum of any other two terms equally distant from the extremes.*

In the series of numbers above, the sum of the first and the last term, of the first but one and the last but one, &c. is 14. And in the other series, the sum of each pair of corresponding terms is $2a+4d$.

To find the sum of *all* the terms in the double series, we have only to observe, that it is equal to the sum of the extremes repeated as many times as there are terms.

The sum of 14, 14, 14, 14, $14=14\times5$.

And the sum of the terms in the other double series is $(2a+4d)\times5$.

But this is *twice* the sum of the terms in the single series. If then we put

$a=$ the first term, $n=$ the number of terms,

$z=$ the last, $s=$ the sum of the terms,

we shall have this equation,

$$s=\frac{a+z}{2}\times n.$$ That is,

439. In an arithmetical progression, *the sum of all the terms is equal to half the sum of the extremes multiplied into the number of terms.*

Prob. What is the sum of the natural series of numbers 1, 2, 3, 4, 5, &c. up to 1000?

Ans. $s=\frac{a+z}{2}\times n=\frac{1+1000}{2}\times1000=500500.$

If in the preceding equation, we substitute for z, its value as given in Art. 436, we have

1. $$s=\frac{2a+(n-1)d}{2}\times n.$$

In this, there are four different quantities, the *first* term of the series, the *common difference*, the *number* of terms, and the sum of the terms; any three of which being given, the fourth may be found. For, by reducing the equation, we have,

2. $$a=\frac{2s-dn^2+dn}{2n},$$ the *first term.*

3. $d=\frac{2s-2an}{n^2-n}$, the *common difference.*

4. $n=\frac{\sqrt{(2a-d)^2+8ds}-2a+d}{2d}$, the number of terms.

Ex. 1. If the first term of an increasing arithmetical series is 3, the common difference 2, and the number of terms 20; what is the sum of the series? Ans. 440.

2. If 100 stones be placed in a straight line, at the distance of a yard from each other; how far must a person travel, to bring them one by one to a box placed at the distance of a yard from the first stone? Ans. 5 miles and 1300 yards.

3. What is the sum of 150 terms of the series $\frac{1}{3}, \frac{2}{3}, 1, \frac{4}{3}, \frac{5}{3}, 2, \frac{7}{3}$, &c.? Ans. 3775.

4. If the sum of an arithmetical series is 1455, the least term 5, and the number of terms 30; what is the common difference? Ans. 3.

5. If the sum of an arithmetical series is 567, the first term 7, and the common difference 2; what is the number of terms? Ans. 21.

6. What is the sum of 32 terms of the series 1, $1\frac{1}{2}$, 2, $2\frac{1}{2}$, 3, &c.? Ans. 280.

7. A gentleman bought 47 books, and gave 10 cents for the first, 30 cents for the second, 50 cents for the third, &c. What did he give for the whole? Ans. 220 dollars, 90 cents.

8. A person put into a charity box, a cent the first day of the year, two cents the second day, three cents the third day, &c. to the end of the year. What was the whole sum for 365 days? Ans. 667 dollars, 95 cents.

440. In the series of *odd* numbers 1, 3, 5, 7, 9, &c. continued to any given extent, the last term is always one less than twice the number of terms.

For $z=a+(n-1)d$. (Art. 435.) But in the proposed series $a=1$, and $d=2$.

The equation, then, becomes $z=1+(n-1)\times 2=2n-1$.

441. In the series of odd numbers, 1, 3, 5, 7, 9, &c. *the sum of the terms is always equal to the square of the number of terms.*

For $s=\frac{1}{2}(a+z)n$. (Art. 439.)

But here $a=1$, and by the last article, $z=2n-1$.

The equation, then, becomes $s=\frac{1}{2}(1+2n-1)n=n^2$.

$$\text{Thus}\quad \left.\begin{array}{r} 1+3=4 \\ 1+3+5=9 \\ 1+3+5+7=16 \end{array}\right\}\text{the square of the number of terms.}$$

442. If there be two ranks of quantities in arithmetical progression, the *sums* or *differences* will also be in arithmetical progression.

For by the addition or subtraction of the corresponding terms, the *ratios* are added or subtracted. (Art. 349.) And by the nature of progression, all the ratios in the series are *equal.* Therefore equal ratios being added to, or subtracted from, equal ratios, the new ratios thence arising will also be equal.

$$\begin{array}{ll} \text{To and from} & 3,\ 6,\ 9,\ 12,\ 15,\ 18,\ 21 \\ \text{Add and subtract} & \underline{2,\ 4,\ 6,\ 8,\ 10,\ 12,\ 14} \\ \text{Sums} & 5,\ 10,\ 15,\ 20,\ 25,\ 30,\ 35 \\ \text{Differences} & 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7 \end{array}\left.\vphantom{\begin{array}{l}a\\a\\a\\a\end{array}}\right\}\text{whose ratio is}\left\{\begin{array}{c} 3 \\ 2 \\ \hline 5 \\ 1 \end{array}\right.$$

443. If all the terms of an arithmetical progression be *multiplied* or *divided* by the same quantity, the products or quotients will be in arithmetical progression.

For by the multiplication or division of the terms, the *ratios* are multiplied or divided; (Art. 348,) that is, equal quantities are multiplied or divided by the given quantity. They will therefore remain equal.

If the series 3, 5, 7, 9, 11, &c. be multiplied by 4;
The prods. will be 12, 20, 28, 36, 44, &c. and if this be div. by 2;
The quots. will be 6, 10, 14, 18, 22, &c.

Problems of various kinds, in arithmetical progression, may be solved, by stating the conditions algebraically, and then reducing the equations.

Prob. 1. Find four numbers in arithmetical progression, whose sum shall be 56, and the sum of their squares 864.

If $x=$ the second of the four numbers,

And $y=$ their common difference:

The series will be $x-y$, x, $x+y$, $x+2y$.

$$\left.\begin{array}{ll}\text{By the conditions,} & (x-y)+x+(x+y)+(x+2y) = 56 \\ \text{And} & (x-y)^2+x^2+(x+y)^2+(x+2y)^2=864\end{array}\right\}$$

$$\left.\begin{array}{ll}\text{That is,} & 4x+2y=56 \\ \text{And} & 4x^2+4xy+6y^2=864\end{array}\right\}$$

Reducing these equations, we have $x=12$, and $y=4$

The numbers required, therefore, are 8, 12, 16, and 20.

Prob. 2. The sum of three numbers in arithmetical progression is 9, and the sum of their cubes is 153. What are the numbers? Ans. 1, 3, and 5.

Prob. 3. The sum of three numbers in arithmetical progression is 15; and the sum of the squares of the two extremes is 58. What are the numbers?

Prob. 4. There are four numbers in arithmetical progression; the sum of the squares of the two first is 34; and the sum of the squares of the two last is 130. What are the numbers? Ans. 3, 5, 7, and 9.

Prob. 5. A certain number consists of three digits, which are in arithmetical progression; and the number divided by the sum of its digits is equal to 26; but if 198 be added to it, the digits will be inverted. What is the number?

Let the digits be equal to $x-y$, x, and $x+y$, respectively. Then the number $=100(x-y)+10x+(x+y)=111x-99y$.

$$\left.\begin{array}{ll}\text{By the conditions,} & \dfrac{111x-99y}{3x}=26 \\ \text{And} & 111x-99y+198=100(x+y)+10x+(x-y)\end{array}\right\}$$

Therefore, $x=3$, $y=1$, and the number is 234.

Prob. 6. The sum of the squares of the extremes of four numbers in arithmetical progression is 200; and the sum of the squares of the means is 136. What are the numbers?

Prob. 7. There are four numbers in arithmetical progression, whose sum is 28, and their continual product 585. What are the numbers?

Geometrical Progression.

444. As arithmetical proportion continued is arithmetical progression, so geometrical proportion continued is geometrical progression.

The numbers 64, 32, 16, 8, 4, are in continued geometrical proportion. (Art. 380.)

In this series, if each preceding term be *divided* by the common ratio, the quotient will be the following term.

$\frac{64}{2}=32$, and $\frac{32}{2}=16$, and $\frac{16}{2}=8$, and $\frac{8}{2}=4$.

If the order of the series be *inverted*, the proportion will still be preserved; (Art. 408,) and the common divisor will become a multiplier. In the series

4, 8, 16, 32, 64, &c. $4\times2=8$, and $8\times2=16$, and $16\times2=32$, &c.

445. *Quantities then are in geometrical progression, when they increase by a common multiplier, or decrease by a common divisor.*

The common multiplier or divisor is called the *ratio*. For most purposes, however, it will be more simple to consider the ratio as always a *multiplier*, either integral or fractional.

In the series 64, 32, 16, 8, 4, the ratio is either 2 a divisor, or $\frac{1}{2}$ a multiplier.

To investigate the properties of geometrical progression, we may take nearly the same course, as in arithmetical progression, observing to substitute continual *multiplication and division*, instead of addition and subtraction. It is evident, in the first place, that,

446. In an ascending geometrical series, each succeeding term is found, by *multiplying the ratio* into the preceding term.

If the first term is a, and the ratio r,

Then $a\times r=ar$, the second term, $ar^2\times r=ar^3$, the fourth,
$ar\times r=ar^2$, the third, $ar^3\times r=ar^4$, the fifth, &c.

And the series is a, ar, ar^2, ar^3, ar^4, ar^5, &c.

447. If the first term and the ratio are the *same*, the progression is simply a series of powers.

If the first term and the ratio are each equal to r,

Then $r\times r=r^2$, the second term, $r^3\times r=r^4$, the fourth,
$r^2\times r=r^3$, the third, $r^4\times r=r^5$, the fifth.

And the series is r, r^2, r^3, r^4, r^5, r^6, &c.

448. In a *descending* series, each succeeding term is found by dividing the preceding term by the ratio, or multiplying by the fractional ratio.

If the first term is ar^6, and the ratio r,

the second term is $\frac{ar^6}{r}$, or $ar^6\times\frac{1}{r}$;

And the series is ar^6, ar^5, ar^4, ar^3, ar^2, ar, a, &c.

If the first term is a, and the ratio r,

The series is $a, \frac{a}{r}, \frac{a}{r^2}, \frac{a}{r^3}$, &c. or a, ar^{-1}, ar^{-2}, &c.

By attending to the series $\overset{1}{a}, \overset{2}{ar}, \overset{3}{ar^2}, \overset{4}{ar^3}, \overset{5}{ar^4}, \overset{6}{ar^5}$, &c. it will be seen that, in each term, the exponent of the power of the ratio, is *one less*, than the number of the term.

If then $a=$ the first term, $r=$ the ratio,
$z=$ the last, $n=$ the number of terms;
we have the equation $z=ar^{n-1}$, that is,

449. In geometrical progression, *the last term is equal to the product of the first, into that power of the ratio whose index is one less than the number of terms.*

When the least term and the ratio are the *same*, the equation becomes $z=rr^{n-1}=r^n$. See Art. 447.

450. Of the four quantities, a, z, r, and n, any *three* being given, the other may be found.*

1. By the last article,

$$z=ar^{n-1}=\text{ the } \textit{last term}.$$

2. Dividing by r^{n-1},

$$\frac{z}{r^{n-1}}=a=\text{ the } \textit{first term}.$$

3. Dividing the 1st by a, and extracting the root,

$$\left(\frac{z}{a}\right)^{\frac{1}{n-1}}=r=\text{ the } \textit{ratio}.$$

By the last equation may be found any number of *geometrical means*, between two given numbers. If $m=$ the number of means, $m+2=n$, the *whole* number of terms. Substituting $m+2$ for n, in the equation, we have

$$\left(\frac{z}{a}\right)^{\frac{1}{m+1}}=r, \text{ the ratio.}$$

When the ratio is found, the means are obtained by continued multiplication.

Prob. 1. Find two geometrical means between 4 and 256.
Ans. The ratio is 4, and the series is 4, 16, 64, 256.

* See Note N.

Prob. 2. Find three geometrical means between $\frac{1}{9}$ and 9.
Ans. $\frac{1}{3}$, 1, and 3.

451. The next thing to be attended to, is the rule for finding the *sum of all the terms.*

If any term, in a geometrical series be multiplied by the ratio, the product will be the succeeding term. (Art. 446.) Of course, if *each* of the terms be multiplied by the ratio, a new series will be produced, in which all the terms except the last will be the same, as all except the first in the other series. To make this plain, let the new series be written under the other, in such a manner, that each term shall be removed one step to the right of that from which it is produced in the line above.

Take, for instance, the series, 2, 4, 8, 16, 32
Multiplying each term by the ratio, we have 4, 8, 16, 32, 64

Here it will be seen at once, that the four last terms in the upper line are the same, as the four first in the lower line. The only terms which are not in *both*, are the *first* of the one series, and the *last* of the other. So that when we subtract the one series, from the other, all the terms except these two will disappear, by balancing each other.

If the given series is $a, ar, ar^2, ar^3, \ldots . ar^{n-1}$
Then mult. by r, we have $ar, ar^2, ar^3, \ldots . ar^{n-1}, ar^n$.

Now let $s=$ the sum of the terms.

Then, $s=a+ar+ar^2+ar^3, \ldots . +ar^{n-1}$,
And mult. by r, $rs= \quad ar+ar^2+ar^3, \ldots . +ar^{n-1}+ar^n$.

Subtracting the first equation from the second, $rs-s=ar^n-a$

And dividing by $(r-1)$, (Art. 124,) $s=\dfrac{ar^n-a}{r-1}$.

In this equation, ar^n is the last term in the new series, and is therefore the product of the ratio into the last term in the *given series.*

Therefore $s=\dfrac{rz-a}{r-1}$, that is,

452. The sum of a series in geometrical progression is found, by multiplying the last term into the ratio, subtracting the first term, and dividing the remainder by the ratio less one.

Prob. 1. If in a series of numbers in geometrical progression, the first term is 6, the last term 1458, and the ratio 3, what is the sum of all the terms?

Ans. $s=\frac{rz-a}{r-1}=\frac{3\times1458-6}{3-1}=2184.$

Prob. 2. If the first term of a decreasing geometrical series is $\frac{1}{2}$, the ratio $\frac{1}{3}$, and the number of terms 5; what is the sum of the series?

The last term$=ar^{n-1}=\frac{1}{2}\times(\frac{1}{3})^4=\frac{1}{162}$.

And the sum of the terms $=\frac{\frac{1}{3}\times\frac{1}{162}-\frac{1}{2}}{\frac{1}{3}-1}=\frac{121}{162}$.

Prob. 3. What is the sum of the series, 1, 3, 9, 27, &c. to 12 terms? Ans. 265720.

Prob. 4. What is the sum of ten terms of the series 1, $\frac{2}{3}$, $\frac{4}{9}$, $\frac{8}{27}$, &c. Ans. $\frac{174075}{59049}$.

453. *Quantities in geometrical progression are proportional to their differences.*

Let the series be a, ar, ar^2, ar^3, ar^4, &c.

By the nature of geometrical progression,

$$a:ar::ar:ar^2:ar^2:ar^3::ar^3:ar^4, \text{ \&c.}$$

In each couplet let the antecedent be subtracted from the consequent, according to Art. 397, 6.

Then $a:ar::ar-a:ar^2-ar::ar^2-ar:ar^3-ar^2$, &c.

That is, the first term is to the second, as the difference between the first and second, to the difference between the second and third; and as the difference between the second and third, to the difference between the third and fourth, &c.

Cor. If quantities are in geometrical progression, their *differences* are also in geometrical progression.

Thus the numbers 3, 9, 27, 81, 243, &c.

And their differences, 6, 18, 54, 162, &c. are in geometrical progression.

454. Several quantities are said to be in *harmonical progression*, when, of any three which are contiguous in the series, the first is to the last, as the difference between the two first, to the difference between the two last. See Art. 409.

Thus the numbers 60, 30, 20, 15, 12, 10, are in harmonical progression.

For $60:20::60-30:30-20$, And $20:12::20-15:15-12$
And $30:15::30-20:20-15$, And $15:10::15-12:12-10$.

Problems in geometrical progression, may be solved, as in other parts of algebra, by the reduction of equations.

Prob. 1. Find three numbers in geometrical progression, such that their sum shall be 14, and the sum of their squares 84.

Let the three numbers be x, y, and z.

By the conditions, $x:y::y:z$, or $xz=y^2$
And $x+y+z=14$
And $x^2+y^2+z^2=84$

Reducing these equations, we find the numbers required to be 2, 4 and 8.

Prob. 2. There are three numbers in geometrical progression whose product is 64, and the sum of their cubes is 584. What are the numbers?

If x be the first term, and y the common ratio; the series will be x, xy, xy^2.

By the conditions, $x\times xy\times xy^2$, or $x^3y^3=64$
And $x^3+x^3y^3+x^3y^6=584$

These equations reduced give $x=2$, and $y=2$.

The numbers required, therefore, are 2, 4 and 8.

Prob. 3. There are three numbers in geometrical progression: The sum of the first and last is 52, and the square of the mean is 100. What are the numbers? Ans. 2, 10, and 50.

Prob. 4. Of four numbers in geometrical progression, the sum of the two first is 15, and the sum of the two last is 60. What are the numbers?

Let the series be x, xy, xy^2, xy^3; and the numbers will be found to be 5, 10, 20, and 40.

Prob. 5. A gentleman divided 210 dollars among three servants, in such a manner, that their portions were in geometrical progression; and the first had 90 dollars more than the last. How much had each?

Prob. 6. There are three numbers in geometrical progression, the greatest of which exceeds the least by 15; and the difference of the squares of the greatest and the least, is to the sum of the squares of all the three numbers as 5 to 7. What are the numbers? Ans. 5, 10, and 20.

Prob. 7. There are four numbers in geometrical progression, the second of which is less than the fourth by 24; and the sum of the extremes is to the sum of the means, as 7 to 3. What are the numbers? Ans. 1, 3, 9, 27.

SECTION XIV.

INFINITES AND INFINITESIMALS.

ART. **455.** THE word *infinite* is used in different senses. The ambiguity of the term has been the occasion of much perplexity. It has even led to the absurd supposition that propositions directly contradictory to each other may be mathematically demonstrated. These apparent contradictions are owing to the fact, that what is proved of infinity when understood in one particular manner, is often thought to be true also, when the term has a very different signification. Tho two meanings are insensibly shifted, the one for the other, so that the proposition which is really demonstrated, is exchanged for another which is false and absurd. To prevent mistakes of this nature, it is important that the different meanings be carefully distinguished from each other.

456. INFINITE, in the highest, and perhaps the most proper sense of the word, *is that which is so great, that nothing can be added to it, or supposed to be added.*

In this sense, it is frequently used in speaking of moral and metaphysical subjects. Thus, by infinite wisdom is meant that which will not admit of the least addition. Infinite power is that which can not possibly be increased, even in supposition. This meaning of infinity is not applicable to the mathematics. That which is the subject of the mathematics is *quantity;* (Art. 1,) such quantity as may be conceived of by the human mind. But no idea can be formed of a quantity

so great that nothing can be supposed to be added to it. In this sense, an *infinite number* is inconceivable. We may increase a number by continual addition, till we obtain one that shall exceed any limits which we please to assign. By this, however, we do not arrive at a number to which nothing can be added; but only at one that is beyond any limits which we have hitherto set. Farther additions may be made to it with the same ease, as those by which it has already been increased so far. It is therefore not infinite, in the sense in which the term has now been explained. It is absurd to speak of the *greatest possible* number. No number can be imagined so great as not to admit of being made greater. We must therefore look for another meaning of infinity, before we can apply it, with propriety, to the mathematics.

457. *A mathematical quantity is said to be infinite, when it is supposed to be increased beyond any determinate limits.*

By determinate limits are meant such as can be distinctly stated.* In this sense, the natural series of numbers, 1, 2, 3, 4, 5, &c. may be said to be infinite. For, if any number be mentioned ever so great, another may be supposed still greater.

The two significations of the word infinite are liable to be confounded, because they are in several points of view the same. The higher meaning includes the lower. That which is so great as to admit of no addition, must be beyond any determinate limits. But the lower does not necessarily imply the higher. Though number is capable of being increased beyond any specified limits; it will not follow, that a number can be found to which no farther additions can be made. The two infinites agree in this, that according to each, the things spoken of are great beyond calculation. But they differ widely in another respect. To the one, nothing can be added. To the other, additions can be made at pleasure.

458. In the mathematical sense of the term, there is no absurdity in supposing *one infinite greater than another.*

We may conceive the numbers	2 2 2 2 2 2 2, &c.
And	4 4 4 4 4 4 4, &c.

to be each extended, so far as to reach round the globe, or to the most distant visible star, or beyond any greater boundary which can be mentioned. But if the two series be equally

* See Note O.

extended, the amount of the one will be *twice* as great as the other, though both be infinite.

So if the series $a+ a^2+ a^3+ a^4+ a^5$, &c.

And $9a+9a^2+9a^3+9a^4+9a^5$, &c.

be extended together beyond any specified limits, one will be *nine* times as great as the other. But it would be absurd to suppose one quantity greater than another, if the latter were already so great that nothing could be added to it.

459. An infinite *number of terms* must not be mistaken for an infinite quantity. The terms may be extended beyond any given limits, when the amount of the whole is a finite quantity, and even a small one. If we take half of a unit; then half of the remainder; half of the remaining half, &c. we shall have the series

$$\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{8}+\tfrac{1}{16}+\tfrac{1}{32}, \text{ \&c.}$$

In which each succeeding term is half of the preceding one. Let the progression be continued ever so far, the sum of all the terms can never exceed a unit. For, by the supposition, there is still a remainder equal to the last term. And this remainder must be added, before the amount of the whole can be equal to a unit.

So $\tfrac{8}{2}+\tfrac{8}{4}+\tfrac{8}{8}+\tfrac{8}{16}+\tfrac{8}{32}+\tfrac{8}{64}$, &c. can never exceed 8.

460. *When a quantity is diminished till it becomes* LESS *than any determinate quantity, it is called an* INFINITESIMAL.

Thus, in a series of fractions, $\tfrac{1}{10}$, $\tfrac{1}{100}$, $\tfrac{1}{1000}$, $\tfrac{1}{10000}$, &c. a unit is first divided into ten parts, then into a hundred, a thousand, &c. One of these parts in each succeeding term is ten times less than in the preceding. If then the progression be continued, a portion of a unit may be obtained less than any specified quantity. This is an infinitesimal, and in mathematical language, is said to be *infinitely small.* By this, however, we are not to understand that it can not be made less. The same process that has reduced it below any limit which we have yet specified, may be continued, so as to diminish it still more. And however far the progression may be carried, we shall never arrive at a point where we must necessarily stop.

461. In the sense now explained, *mathematical quantity* may be said to be *infinitely divisible;* that is, it may be supposed to be so divided, that the *parts* shall be *less* than any

determinate quantity, and the *number* of parts *greater* than any given number.

In the series, $\frac{1}{10}$, $\frac{1}{100}$, $\frac{1}{1000}$, $\frac{1}{10000}$, &c. a unit is divided into a greater and greater number of parts, till they become infinitesimals, and the number of them infinite, that is, such a number as exceeds any *given* number. But this does not prove that we can ever arrive at a division in which the parts shall be the *least possible* or the *number* of parts the *greatest possible.*

462. One infinitesimal may be *less* than another.

$$\left.\begin{array}{ll}\text{The series} & \frac{6}{10},\ \frac{6}{100},\ \frac{6}{1000},\ \frac{6}{10000},\ \&c. \\ \text{And} & \frac{3}{10},\ \frac{3}{100},\ \frac{3}{1000},\ \frac{3}{10000},\ \&c.\end{array}\right\}$$

may be carried on together, till the last term in each becomes infinitely small; and yet one of these terms will be only *half* as great as the other. For the denominators being the same, the fractions will be as their numerators, (Art. 364, cor. 1,) that is, as $6 : 3$, or $2 : 1$.

Two quantities may also be divided, each into an infinite number of parts, using the term infinite in the mathematical sense, and yet the parts of one be more numerous than those of the other.

$$\left.\begin{array}{ll}\text{The series} & \frac{1}{10},\ \frac{1}{100},\ \frac{1}{1000},\ \frac{1}{10000},\ \&c. \\ \text{And} & \frac{1}{40},\ \frac{1}{400},\ \frac{1}{4000},\ \frac{1}{40000},\ \&c.\end{array}\right\}$$

may both be infinitely extended; and yet a unit in the last series, is divided into four times as many parts as in the first. But if, by an infinite number of parts were meant, such a number as could not be increased, it would be absurd to suppose the divisions of any quantity to be still more numerous.*

463. For all *practical* purposes, an infinitesimal may be considered as absolutely nothing. As it is less than any determinate quantity, it is lost even in numerical calculations. In algebraic processes, a term is often rejected as of no value, because it is infinitely small.

It is frequently expedient to admit into a calculation, a small error, or what is suspected to be an error. It may be difficult either to avoid the objectionable part, or to ascertain its exact value, or even to determine, without a long and tedious process, whether it is really an error or not. But if it can be shown to be infinitely small, it is of no account in practice, and may be retained or rejected at pleasure.

* See Note P.

It is impossible to find a decimal which shall be exactly equal to the vulgar fraction $\frac{1}{3}$. Dividing the numerator by the denominator, we obtain in the first place $\frac{3}{10}$. This is nearly equal to $\frac{1}{3}$. But $\frac{33}{100}$ is nearer, $\frac{333}{1000}$ still nearer, &c.

The error, in the first instance, is $\frac{1}{30}$.

For $\frac{3}{10}+\frac{1}{30}=\frac{9}{30}+\frac{1}{30}=\frac{10}{30}=\frac{1}{3}$.

In the same manner it may be shown, that

the difference between $\begin{cases} \frac{1}{3} \text{ and } .33, \text{ is } \frac{1}{300}, \\ \frac{1}{3} \text{ and } .333, \text{ is } \frac{1}{3000}, \text{ \&c.} \end{cases}$

If the decimal be supposed to be extended beyond any assignable limit, the difference still remaining will be infinitely small. As this error is less than any given quantity, it is of no account, and may be considered in calculation as nothing.

464. From the preceding example it will be seen, that a quantity may be continually *coming nearer* to another, and yet *never reach it*. The decimal 0.3333333, &c. by repeated additions on the right, may be made to approximate continually to $\frac{1}{3}$, but can never exactly equal it. A difference will always remain, though it may become infinitely small.

When one quantity is thus made to approach continually to another, without ever passing it; the latter is called a *limit* of the former. The fraction $\frac{2}{3}$ is a limit of the decimal 0.666, &c. indefinitely continued.

465. Though an infinitesimal is of no account of *itself*, yet its effect on other quantities is not always to be disregarded. When it is a *factor* or *divisor*, it may have an important influence. It is necessary, therefore, to attend to the relations which infinites, infinitesimals, and finite quantities have to each other. As an infinitesimal is less than any assignable quantity, as it is next to nothing, and, in practice, may be considered as nothing, it is frequently represented by 0.

An infinite quantity is expressed by the character ∞.

466. As an infinite quantity is incomparably greater than a finite one, the alteration of the former, by an *addition* or *subtraction* of the latter, may be disregarded in calculation. A single grain of sand is greater in comparison with the whole earth, than any finite quantity in comparison with one which is infinite. If therefore infinite and finite quantities are connected by the sign + or −, the latter may be rejected as of no comparative value. For the same reason, if finite

quantities and infinitesimals are connected by + or −, the latter may be expunged.

467. But if an infinite quantity be *multiplied* by one which is finite, it will be as many times increased as any other quantity would, by the same multiplier.

If the infinite series 2 2 2 2 2 2, &c. be multiplied by 4;

The product will be 8 8 8 8 8 8, &c. four times as great as the multiplicand. See Art. 458.

468. And if an infinite quantity be *divided* by a finite quantity, it will be altered in the same manner as any other quantity.

If the infinite series 6 6 6 6 6 6 6 6, &c. be divided by 2;

The quotient will be 3 3 3 3 3 3 3 3, &c. half as great as the dividend.

469. If a *finite quantity* be multiplied by an *infinitesimal*, the product will be an infinitesimal; that is, putting z for a finite quantity, and 0 for an infinitesimal, (Art. 465,)

$$z \times 0 = 0.$$

If the multiplier were a *unit*, the product would be equal to the multiplicand. (Art. 85.) If the multiplier is less than a unit, the product is proportionally less. If then the multiplier is *infinitely* less than a unit, the product must be infinitely less than the multiplicand, that is, it must be an infinitesimal. Or, if an infinitesimal be considered as absolutely nothing, then the product of z into nothing is nothing. (Art. 106.)

470. On the other hand, if a finite quantity be *divided* by an infinitesimal, the quotient will be infinite.

$$\frac{z}{0} = \infty.$$

For, the less the divisor, the greater the quotient. If then the divisor be *infinitely* small, the quotient will be infinitely great. In other words, an infinitesimal is contained an infinite number of times in a finite quantity. This may, at first, appear paradoxical. But it is evident, that the quotient must increase as the divisor is diminished.

Thus $6 \div 3 = 2$, $\quad 6 \div 0.03 = 200$,

$6 \div 0.3 = 20$, $\quad 6 \div 0.003 = 2000$, &c.

If then the divisor be reduced, so as to become *less* than any assignable quantity, the quotient must be *greater* than any assignable quantity.

471. If a finite quantity be divided by an infinite quantity, the quotient will be an infinitesimal.

$$\frac{z}{\infty}=0.$$

For the greater the divisor, the less the quotient. If then, while the dividend is finite, the divisor be infinitely great, the quotient will be infinitely small.

472. It is evident from Art. 469, that $\frac{0}{z}=0$, that is, if an infinitesimal be divided by a finite quantity, the quotient will be an infinitesimal.

It must not be forgotten, that the expressions *infinitely great* and *infinitely small*, are, all along, to be understood in the *mathematical* sense according to the definitions in Arts. 457, and 460.

473. The expression $\frac{0}{0}$ occurs frequently, in the higher departments of the mathematics, particularly in the differential and integral calculus; which treat principally of *variable* quantities, and in which, these are often considered as reduced to infinitesimals. Even in this state, there may be a definite ratio between them. This, however, is not indicated by the expression $\frac{0}{0}$. If all infinitesimals were *equal*, the quotient of one divided by another would invariably be unity. But as one of them may be greater than another, (Art. 458,) the numerator of the fraction $\frac{0}{0}$ may be greater or less than the denominator. The value of the expression taken abstractly, without reference to its *origin* in particular cases, is said to be *indeterminate*. But when we trace it back, to the quantities of assignable magnitude from which it has been derived, we may discover the ratio which the dividend bears to the divisor: to take a very simple example,

$$\frac{a^2-b^2}{a-b}=a+b. \quad \text{(Art. 130.)}$$

Considering b as a variable quantity, if it be made equal to a, we have $\frac{0}{0}=2a$.

474. The expression $\frac{0}{0}$ is frequently derived from a fraction which has a *common factor* in the numerator and denominator. When this, on a particular supposition, becomes 0, it reduces to 0 the expressions into which it entered as a factor. (Art. 469.) If it be removed by division, the definite ratio of the numerator of the fraction to the denominator may be ascertained. Thus in the example just given, if we suppose $b=a$, (a^2-b^2) and $a-b$ are each reduced to 0. But dividing by the common factor $(a-b)$ we have $a+b$, or $2a$, as before.

Though dividing the terms of a fraction by a common factor does not *alter* the ratio between them, (Art. 145,) yet it may enable us to *discover* the ratio which still continues, even in the expression $\frac{0}{0}$.

Ex. 2. The fraction $\frac{a^3-b^3}{a^2-b^2}$ becomes $\frac{0}{0}$, when b is made equal to a. But if it be divided by the common factor $a-b$ (Art. 130,) the quotient is $\frac{a^2+ab+b^2}{a+b}$, which, when $b=a$ becomes $\frac{3a^2}{2a}=\frac{3a}{2}$, a definite quantity.

Ex. 3. The fraction $\frac{x^4+ax^3-9a^2x^2+11a^3x-4a^4}{x^4-ax^3-3a^2x^2+5a^3x-2a^4}$ becomes $\frac{0}{0}$ when $x=a$. But if the numerator and denominator be divided by the common factor $(x-a)^3$, the fraction becomes $\frac{x+4a}{x+2a}$ which $=\frac{5}{3}$ when $x=a$.

475. In other cases, a fraction which, on a particular supposition, gives $\frac{0}{0}$, when reduced to its lowest terms, may retain the 0, either in the numerator or the denominator; so that its value will be either 0 or ∞.

Ex. 4. The fraction $\frac{3a^3-10a^2x+4ax^2+8x^3}{a^2-4x^2}$ becomes $\frac{0}{0}$, on the supposition that $x=\frac{1}{2}a$. But if it be divided by the

common factor $a-2x$ which $=0$, it becomes $\frac{3a^2-4ax-4x^2}{a+2x}$ which $=\frac{0}{4x}$, an infinitesimal, (Art. 472.)

Ex. 5. The fraction $\frac{3a^2-4ax-4x^2}{a^3-2a^2x-4ax^2+8x^3}$ becomes $\frac{0}{0}$ when $x=\frac{1}{2}a$. If it be divided by the common factor $a-2x$, it comes $\frac{3a+2x}{a^2-4x^2}$ which $=\frac{8x}{0}$, an infinite quantity, (Art. 470.)

SECTION XV.

COMMON MEASURE, AND MULTIPLE, PERMUTATIONS AND COMBINATIONS.

Art. **476.** The Greatest Common Measure of two quantities may be found by the following rule;

Divide one of the quantities by the other, and the preceding divisor by the last remainder, till nothing remains; the last divisor will be the greatest common measure.

The algebraic letters are here supposed to stand for whole numbers. In the demonstration of the rule, the following principles must be admitted.

1. Any quantity measures *itself*, the quotient being 1.

2. If two quantities are respectively measured by a third, their *sum* or *difference* is measured by that third quantity. If b and c are each measured by d, it is evident that $b+c$, and $b-c$ are measured by d. Connecting them by the sign $+$ or $-$, does not affect their capacity of being measured by d.

Hence, if b is measured by d, then by the preceding proposition, $b+d$ is measured by d.

3. If one quantity is measured by another, any *multiple* of the former is measured by the latter. If b is measured by d, it is evident that $b+b$, $3b$, $4b$, nb, &c. are measured by d.

Now let $D=$ the greater, and $d=$ the less of two algebraic quantities, whether simple or compound. And let the process of dividing, according to the rule be as follows:

$$\begin{array}{l} d)D(q \\ \underline{\;dq\;} \\ \quad r)d(q' \\ \quad \underline{\;rq'\;} \\ \qquad r')r(q'' \\ \qquad \underline{\;r'q''\;} \\ \qquad\quad \underline{o} \end{array}$$

In which q, q', q'', are the *quotients*, from the successive divisions; and r, r', and o the *remainders*. And as the dividend is equal to the product of the divisor and quotient added to the remainder,

$$D=dq+r, \qquad \text{and } d=rq'+r'.$$

Then, as the last divisor r' measures r the remainder being o,

it measures (2, and 3,) $rq'+r'=d$,

and measures $dq+r=D$.

That is, the last divisor r' is a common measure of the two given quantities D and d.

It is also their *greatest* common measure. For every common measure of D and d, is also (3, and 2,) a measure of $D-dq=r$; and every common measure of d and r, is also a measure of $d-rq'=r'$. But the *greatest* measure of r' is *itself*. This, then, is the greatest common measure of D and d.

The demonstration will be substantially the same, whatever be the number of successive divisions, if the operation be continued till the remainder is nothing.

To find the greatest common measure of *three* quantities; first find the greatest common measure of two of them, and then, the greatest common measure of this and the third quantity. If the greatest common measure of D and d be r', the greatest common measure of r' and c, is the greatest common measure of the three quantities D, d, and c. For *every* measure of r', is a measure of D and d; therefore the *greatest* common measure of r' and c, is also the greatest common measure of D, d, and c.

The rule may be extended to any number of quantities.

477. There is not much occasion for the preceding operations, in finding the greatest common measure of *simple* algebraic quantities. For this purpose, a glance of the eye will generally be sufficient. In the application of the rule to *compound* quantities, it will frequently be expedient to reduce the divisor, or enlarge the dividend, in conformity with the following principle:

The greatest common measure of two quantities is not altered, by multiplying or dividing either of them by any quantity which is not a divisor of the other, and which contains no factor which is a divisor of the other.

The common measure of ab and ac is a. If either be *multiplied* by d, the common measure of abd, and ac, or of ab and acd, is still a. On the other hand, if ab and acd are the given quantities, the common measure is a; and if acd be *divided* by d, the common measure of ab and ac is a.

Hence in finding the common measure by division, the divisor may often be rendered more simple, by dividing it by some quantity, which does not contain a divisor of the dividend. Or the dividend may be *multiplied* by a factor, which does not contain a measure of the divisor. This may be necessary, when the first term of the divisor is not contained in the first term of the dividend.

Ex. 1. Find the greatest common measure of

$$6a^2+11ax+3x^2,\text{ and } 6a^2+7ax-3x^2.$$

$$
\begin{array}{l}
6a^2+7ax-3x^2)6a^2+11ax+3x^2(1 \\
\qquad\qquad\qquad\quad \underline{6a^2+\ 7ax-3x^2} \\
\text{Dividing by}\quad 2x)4ax+6x^2 \\
\qquad\qquad\qquad\qquad 2a+3x)6a^2+7ax-3x^2(3a-x \\
\qquad\qquad\qquad\qquad\qquad\qquad \underline{6a^2+9ax} \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad -2ax-3x^2 \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{-2ax-3x^2} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad * \quad *
\end{array}
$$

After the first division here, the remainder is divided by $2x$, which reduces it to $2a+3x$. The division of the preceding divisor by this, leaves no remainder. Therefore $2a+3x$ is the common measure required.

2. What is the greatest common measure of x^3-b^2x, and $x^2+2bx+b^2$? Ans. $x+b$.

3. What is the greatest common measure of $cx+x^2$, and a^2c+a^2x? Ans. $c+x$.

4. What is the greatest common measure of $3x^3-24x-9$, and $2x^3-16x-6$? Ans. x^3-8x-3.

5. What is the greatest common measure of a^4-b^4, and $a^5-b^2a^3$? Ans. a^2-b^2.

6. What is the greatest common measure of x^2-1, and $xy+y$? Ans. $x+1$.

7. What is the greatest common measure of x^3-a^3, and x^4-a^4?

8. What is the greatest common measure of $a^2-ab-2b^2$, and $a^2-3ab+2b^2$?

9. What is the greatest common measure of a^4-x^4, and $a^3-a^2x-ax^2+x^3$?

10. What is the greatest common measure of a^3-ab^2, and $a^2+2ab+b^2$?

x

Least Common Multiple.

478. To find the least common multiple of two or more quantities,

Let $m=$ the least common multiple of a and b.
$x=$ their greatest common measure.
$p=$ the number of times a is contained as a factor in m.
$q=$ the number of times b is contained in m.

Then $ap=m$, and $bq=m$. Therefore, $ap=bq$.

Dividing both by bp, we have $\frac{a}{b}=\frac{q}{p}$. And $\frac{q}{p}$ is the fraction $\frac{a}{b}$ in its lowest terms. If not, let $\frac{q'}{p'}$ be this fraction in its lowest terms, p' and q' being less than p and q. Then as $\frac{q'}{p'}=\frac{a}{b}$, $ap'=bq'$, (Art. 186.) And as ap' is divisible by a, its equal bq' must also be divisible by a, as well as by b. Therefore bq' is a common multiple of a and b, a multiple less than bq, or its equal m; that is, *less than the least* common multiple of a and b, which is impossible.

Now the number which, by division, reduces a fraction to its lowest terms, is the greatest *common measure* of the numerator and denominator. (Art. 149.) We have then $\frac{a}{x}=q$, and $\frac{b}{x}=p$.

As $m=ap$, substituting $\frac{b}{x}$ for p, we have, $m=\frac{ab}{x}$.

That is, *The least common multiple of two numbers is equal to their* PRODUCT *divided by their greatest* COMMON MEASURE.

PERMUTATIONS.

479. The different results obtained, by varying the *order* in which a given number of letters or things may be written or placed, one after another, are called *permutations*.

Thus ab and ba are permutations of the letters a and b. And abc, acb, bac, bca, cab, and cba, are permutations of a, b, and c; each of the three letters being on the left, in the middle, and on the right, of two results.

The *number* of the permutations of *two* letters is evidently two, ab and ba.

To find the number of permutations of *three* letters, a, b, and c; observe

That a, placed before each of the permutations of b and c gives abc, acb

b, before those of a and c gives bac, bca

c, before those of a and b gives cab, cba

The whole number is 3×2, = the whole number of letters, multiplied by the permutations of the *next less* number.

Or thus, In each of the permutations of a and b, the additional letter c may have three positions, before, between, and after, the other two. Therefore all the permutations of three letters $=3\times2$, as before.

If there be four letters, $abcd$, each of these may be placed successively before the six permutations of three letters; making the whole number from the four letters, $4\times3\times2=24$.

Or, the additional letter d may have four positions in each of the permutations of the other three. It may stand before each of the three, and after them all.

So if n be *any* number of letters, and Q the permutations of $n-1$ letters, the permutations of n letters $=n\times Q$. For each of the n letters may be placed before the several permutations of $n-1$ letters.

In this way, we pass from the permutations of a given number of letters, to those of the next greater number. It follows, that the number of permutations corresponding to each of the natural series of numbers 1, 2, 3, 4, &c. = the *product* of the factors 1 . 2 . 3 . 4, &c.

$$\text{The permutations}\begin{cases}\text{of } \textit{three} \text{ letters are } 1.2.3 & = 6\\ \text{of } \textit{four} \text{ are } 1.2.3.4 & = 24\\ \text{of } \textit{five} \text{ are } 1.2.3.4.5 & =120\\ \text{of } n \text{ letters are } 1.2.3\ldots\text{up to } n.\end{cases}$$

Or, reversing the order, $n(n-1)(n-2)(n-3)\ldots 3.2.1.$

480. In the preceding cases, each permutation, when the solution is completed, contains *all* the letters given, in the problem to be solved. But we have frequent occasion to determine how many pairs, triplets, or larger sets, can be taken from a greater number of letters or things; for instance, how many permutations can be made from four letters, *a, b, c,* and *d,* taking them *two and two, ab, ac, ad,* &c. or *three and three, abc, abd, bcd,* &c.

These classes of results are sometimes termed *arrangements,* to distinguish them from the permutations which contain *all* the given letters.

Letters taken *singly,* or one by one, are also called arrangements, or permutations.

To find the number of permutations of *four* letters, taken in *couplets,* or sets of two each.

Of the four letters, *a b c d,* three at a time may be *reserved,* to be annexed to the remaining one.

$$\text{Annexing}\begin{Bmatrix} b, c, \text{ and } d \text{ to } a\\ a, c, \text{ and } d \text{ to } b\\ a, b, \text{ and } d \text{ to } c\\ a, b, \text{ and } c \text{ to } d\end{Bmatrix}\text{ we have }\begin{cases} ab & ac & ad\\ ba & bc & bd\\ ca & cb & cd\\ da & db & dc\end{cases}$$

The number of couplets $=4\times3=12=$ the number of single letters multiplied by the three which are reserved at a time.

To find the number of permutations of four letters taken in triplets, or sets of three each.

First find, as above, the number of *couplets* formed from the four letters. As each one of these sets, taken by itself, contains only two of the four given letters, the other two are, of course, excluded from it; and are said to be *reserved,* to

be annexed to the couplet, in forming triplets. In this way, we avoid *repeating* any letter, in any one of the sets.

If to *ab*, the first of the couplets above, there be annexed successively the two letters which are not contained in it, we have the triplets *abc* and *abd*. If to *ac*, the second couplet, there be annexed the two letters which are not contained in *that*, we have *acb* and *acd*.

Proceeding in this manner with the other sets, we obtain twelve pairs of triplets, each differing from the others, either in some of the letters, or in the order in which they stand. The whole number of triplets is equal to the product of the number of couplets into the number of letters reserved; $12 \cdot 2 = 4 \cdot 3 \cdot 2$.

481. To apply this mode of calculation to *any* number of letters or things;

Let $n=$ the whole number to be arranged.

And $r=$ the number required to be taken in a set.

Then will $r-1=$ the number in the *next less set.*

And $n-(r-1)$ or $n-r+1=$ the *difference* between the whole number of letters, and the number in the lesser set. This difference then is equal to the number of letters *reserved* from one of the lesser sets.

To obtain then the number of permutations from n letters $a\ b\ c \ldots$ to l, taken in sets of r and r, supposing the permutations taken in sets of $r-1$ to be already known;

Let $P=$ the permutations of n letters in sets of $r-1$.

Then to each of these, the reserved letters, whose number $=n-r+1$, may be applied. Therefore, the whole number of permutations of n letters, in sets of r and r, is

$$P \times (n-r+1)$$

That is, *It is equal to the product of the number of permutations in sets of $r-1$, into the number of letters reserved.*

482. To obtain the number of permutations in succession, as we pass from the smaller sets to the larger, observe that

If $r=1$, then

If $r=2$, $P=n$

If $r=3$, $P=n(n-1)$

If $r=4$, $P=n(n-1)(n-2)$

If $r=5$, $P=n(n-1)(n-2)(n-3)$

$$\text{And } n-r+1=\begin{cases} n \\ n-1 \\ n-2 \\ n-3 \\ n-4 \end{cases}$$

Bearing then in mind that, when the letters are taken *singly*, the permutations are equal to the number of letters, the permutations of n letters, being equal to

$$P\times(n-r+1)$$

If $r=1$, are n

If $r=2$, $n(n-1)$

If $r=3$, $n(n-1)(n-2)$

If $r=4$, $n(n-1)(n-2)(n-3)$

If $r=5$, $n(n-1)(n-2)(n-3)(n-4)$.

From the manner in which the larger of these formulas are derived from the smaller, and from the fact that, in each, the first factor is n, and the others regularly decreasing by 1, till the last is equal to $n-r+1$, it is evident that, for still larger sets, the general formula is

$$n(n-1)(n-2)(n-3)\ldots(n-r+1).$$

If $r=n$, that is, if the permutations are in *sets equal to the whole number of letters*, then $n-r+1$ becomes 1, and the formula is

$n(n-1)(n-2)(n-3)\ldots3.2.1$, the same as in Art. 479.

Ex. 1. How many permutations can be made of the letters in the word chemistry?

Ans. $1.2.3.4.5.6.7.8.9=362880$.

2. How many changes can be rung with ten bells?

3. How many arrangements can be made with 7 letters, taken 4 and 4 together? Ans. $7.6.5.4=840$.

4. How many arrangements can be made with 15 letters, taken 6 and 6?

Combinations.

483. A combination, in algebra, is a collection of letters or things, taken without regard to the *order* in which they are placed. Combinations differ from each other, in at least one of the letters of which they are composed. A given set of letters constitutes only one combination, however various may be the order in which they are arranged. But a single combination contains as many *permutations*, as the changes which it admits in the order of its letters. Thus the six permutations *abc acb bac bca cab cba* are all composed of the same letters, differing in their arrangement only.

Where there are several combinations, it is evident that the whole number of permutations of n letters, is equal to the number of combinations, *multiplied* into the number of permutations in each of the combinations. Therefore, the number of *combinations* of n letters is equal to the whole number of *permutations*, *divided* by the number of permutations in *one* combination.

To find, then, the number of combinations of n letters, taken in sets of r letters at a time,

Let C=the number of *combinations* of n letters in sets of r and r.

X=the number of *permutations* of n letters in sets of r and r.

Y=the number of permutations of r letters.

Then $X=C\times Y$. Or, $C=\frac{X}{Y}$.

Let P= the number of permutations of n letters in sets of $r-1$ as in Art. 481.

Q= the number of permutations of $r-1$ letters, as in Art. 479.

Then $\frac{P}{Q}$= the number of *combinations* in sets of $r-1$.

By Art. 482, $X=P(n-r+1)$.
By Art. 479, $Y=Q\times r$.
Therefore, $C=\frac{P}{Q}\times\frac{n-r+1}{r}$.

To expand this last formula, passing from the smaller numbers to the larger, take the values of P for successive numbers from the list in Art. 482, and the corresponding values of Q from Art. 479, and we have,

If $r=2$, $P=n$	And if $r=2$, $Q=1$
If $r=3$, $P=n(n-1)$	3, $Q=1.2$
If $r=4$, $P=n(n-1)(n-2)$	4, $Q=1.2.3$
If $r=5$, $P=n(n-1)(n-2)(n-3)$	5, $Q=1.2.3.4$

Therefore, the formula $\frac{P}{Q}\times\frac{n-r+1}{r}$ becomes as follows,

If $r=2$, $\frac{n(n-1)}{1\,.\,2.}$

If $r=3$, $\frac{n(n-1)}{1\,.\,2.}\frac{(n-2)}{3.}$

If $r=4$, $\frac{n(n-1)(n-2)(n-3)}{1.\ 2.\ \ 3.\ \ 4.}$

If $r=5$, $\frac{n(n-1)(n-2)(n-3)(n-4)}{1.\ 2.\ \ 3.\ \ 4.\ \ 5.}$

As the right hand factor, in each case, $=\frac{n-r+1}{r}$, the following is the *general* formula for the number of combinations of n letters taken r and r.

$$\frac{n(n-1)(n-2)(n-3)}{1.\ 2.\ \ 3.\ \ 4.}\cdots\frac{n-r+1}{r}.$$

Ex. 1. What is the number of combinations of 10 letters taken 4 and 4? Ans. $\frac{10.9.8.7}{1.2.3.4}=210.$

2. What is the number of combinations of 9 letters taken 3 and 3.

SECTION XVI.

INVOLUTION AND EXPANSION OF BINOMIALS.

Art. **484.** The manner in which a binomial, as well as any other compound quantity, may be involved by repeated multiplications, has been shown in the section on powers. (Art. 230.) But when a high power is required, the operation becomes long and tedious.

This has led mathematicians to seek for some general principle, by which the involution may be more easily and expeditiously performed. We are chiefly indebted to Sir Isaac Newton for the method which is now in common use. It is founded on what is called the *Binomial Theorem*, the invention of which was deemed of such importance to mathematical investigation, that it is engraved on his monument in Westminster Abbey.

485. If the binomial root be $a+b$, we may obtain, by multiplication, the following powers. (Art. 230.)

$(a+b)^2=a^2+2ab+b^2$
$(a+b)^3=a^3+3a^2b+3ab^2+b^3$
$(a+b)^4=a^4+4a^3b+6a^2b^2+4ab^3+b^4$
$(a+b)^5=a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5$, &c.

By attending to this series of powers, we shall find, that the *exponents* preserve an invariable order through the whole. This will be very obvious, if we take the exponents by themselves, unconnected with the letters to which they belong.

In the square, the exponents $\begin{cases} \text{of } a \text{ are } 2, 1, 0 \\ \text{of } b \text{ are } 0, 1, 2 \end{cases}$

In the cube, the exponents $\begin{cases} \text{of } a \text{ are } 3, 2, 1, 0 \\ \text{of } b \text{ are } 0, 1, 2, 3 \end{cases}$

In the 4th power, the exponents $\begin{cases} \text{of } a \text{ are } 4, 3, 2, 1, 0 \\ \text{of } b \text{ are } 0, 1, 2, 3, 4 \end{cases}$

&c.

Here it will be seen at once, that the exponents of a in the *first* term, and of b in the *last*, are each equal to the index of the power; and that the *sum* of the exponents of the two letters is in every term the same.

It is farther to be observed that the exponents of a regularly *decrease* to 0, and that the exponents of b *increase* from 0. That this will universally be the case, to whatever extent the involution may be carried, will be evident, if we consider, that in raising from any power to the next, *each term* is multiplied both by a and by b.

If the exponents, before the multiplication, increase and decrease by 1, and if the multiplication adds 1 to each, it is evident they must still increase and decrease in the same manner as before.

486. If then $a+b$ be raised to a power whose exponent is n,

The exponents of a will be $n, n-1, n-2, \ldots .2, \quad 1, 0$;

And the exponents of b will be $0, 1, \quad 2, \ldots . n-2, n-1, n$.

The terms in which a power is expressed, consist of the *letters* with their *exponents* and the *co-efficients*. Setting aside the co-efficients for the present, we can determine, from the preceding observations, the letters and exponents of any power whatever.

Thus the eighth power of $a+b$, when written without the co-efficients, is

$$a^8+a^7b+a^6b^2+a^5b^3+a^4b^4+a^3b^5+a^2b^6+ab^7+b^8.$$

And the nth power of $a+b$ is,

$$a^n+a^{n-1}b+a^{n-2}b^2\ldots\ldots a^2b^{n-2}+ab^{n-1}+b^n.$$

Of the two letters in a term, the first is called the *leading* quantity, and the other the *following* quantity. In the examples which have been given in this section, a is the leading quantity, and b the following quantity.

487. The *number* of terms is greater by 1, than the index of the power. For if the index of the power is n, a has, in different terms, every index from n down to 1; and there is one additional term which contains only b. Thus,

The square has 3 terms,	The 4th power, 5,
The cube 4,	The 5th power, 6, &c.

488. The next step is to find the *co-efficients*. This part of the subject is more complicated. There is, however, an easy method of calculating the numerical co-efficients of the powers of a binomial, to some extent at least, by actual multiplication. As the co-efficients of $x+a$ are 1 and 1, if these are involved by themselves, the multiplier is continually a unit. In arranging the terms of the powers of a binomial, according to the exponents of the two letters, (Art. 486,) the particular product, from multiplying by the second letter, is carried one term farther to the right, than the product from multiplying by the first. If this arrangement be adopted, in involving $1+1$, the multiplication will be effected by *addition* merely, and the co-efficients will correspond with the terms obtained by involving the letters.

	$1+1$	
	$\quad 1+1$	
The square	$1+2+1$	Whose sum is $4=2^2$
	$\quad 1+2+1$	
The cube	$1+3+3+1$	$8=2^3$
	$\quad 1+3+3+1$	
The 4th power	$1+4+6+4+1$	$16=2^4$
	$\quad 1+4+6+4+1$	
The 5th power	$1+5+10+10+5+1$	$32=2^5$
	$\quad 1+5+10+10+5+1$	
The 6th power	$1+6+15+20+15+6+1$	$64=2^6$

In the same manner, if the co-efficients of the terms of *any* power of a binomial be given, the co-efficients of higher powers may be easily found.

The results thus obtained are the co-efficients, in the several terms of the powers of $x+a$. The *letters* are easily arranged, according to the law in Arts. 485, 6. If then, to the several terms, we prefix their corresponding co-efficients, we have the complete power of the binomial. Thus

$$(x+a)^6=x^6+6x^5a+15x^4a^2+20x^3a^3+15x^2a^4+6xa^5+a^6.$$

489. By recurring to the numbers in Art. 488, it will be seen, that the co-efficients first *increase*, and then *decrease*, at the same rate; so that they are equal, in the first term, and the last, in the second and last but one, in the third and last but two; and universally, in any two terms equally distant from the extremes. The reason of this is, that $(a+b)^n$ is the same as $(b+a)^n$; and if the order of the terms in the binomial root be changed, the whole series of terms in the power will be inverted.

It is sufficient, then, to find the co-efficients of *half* the terms. These repeated will serve for the whole.

490. In any power of $(a+b)$, the sum of the co-efficients is equal to the number 2 raised to that power. See the list of co-efficients in Art. 488. The reason of this is, that, according to the rules of multiplication, when any quantity is involved, the *letters* are multiplied into each other, and the *co-efficients* into each other. Now the co-efficients of $a+b$ being $1+1=2$, if these be involved, a series of the powers of 2 will be produced.

491. To determine the *law* which the co-efficients of *any* term of a binomial uniformly observe, multiply the factors $x+a$, $x+b$, $x+c$, &c. writing the several co-efficients of the same power of x one under the other.

The product of $x+a$ into $x+b$ is

$$x^2 \left.\begin{matrix}+a\\+b\end{matrix}\right\} x+ab. \quad \text{This into } x+c \text{ gives}$$

$$x^3 \left.\begin{matrix}+a\\+b\\+c\end{matrix}\right\} x^2 \left.\begin{matrix}+ab\\+ac\\+bc\end{matrix}\right\} x+abc. \quad \text{This into } x+d \text{ gives}$$

$$x^4\left.\begin{matrix}+a\\+b\\ \\+c\\+d\end{matrix}\right\}x^3\left.\begin{matrix}+ab\\+ac\\+bc\\ \\+ad\\+bd\\+cd\end{matrix}\right\}x^2\left.\begin{matrix}+abc\\+abd\\ \\+acd\\+bcd\end{matrix}\right\}x+abcd.$$

It will be seen that, in each term of these several products, the number of letters in one combination is equal to the number of the *preceding terms*. Also,

In the *first* term, the co-efficient of x is 1.

In the *second*, it is the sum of all the other letters of the binomial factors, taken *singly*.

In the *third*, it is the sum of their products, taken *two and two*.

In the *fourth*, it is the sum of their products, taken *three and three*.

The *last* term is the product of *all* these letters.

And as each multiplication introduces a new letter into the several combinations which constitute the co-efficients of x, the relation of each term to the preceding is such, that if m be the number of any term, and r be the number of the terms which *precede* it, the co-efficient of x in this mth term, is the sum of the several combinations of the other letters of the binomial factors taken r and r, or $m-1$ and $m-1$. To shew this,

492. Let A, B, C, &c. be respectively equal to the different co-efficients of x, in the product of several binomial factors. This product, for n binomial factors, will be

$$x^n+Ax^{n-1}+Bx^{n-2}+Cx^{n-3}\ldots Nx^{n-r+1}+Rx^{n-r}\ldots+U.$$

As in the successive terms, the exponent of x decreases by 1, the term in which this exponent is $n-r$ must have r terms before it; and the exponent of the term immediately preceding must be $n-r+1$.

Multiplying the given product by an *additional* binomial factor $x+h$, we have

$$x^{n+1}\left.\begin{matrix}+A\\+h\end{matrix}\right\}x^n\left.\begin{matrix}+B\\+Ah\end{matrix}\right\}x^{n-1}\left.\begin{matrix}+C\\+Bh\end{matrix}\right\}x^{n-2}\ldots\left.\begin{matrix}R\\Nh\end{matrix}\right\}x^{n-r+1}\ldots+Uh,$$

which is the product of $n+1$ binomial factors,

As multiplying by x affects not the co-efficients, but, in each term, adds 1 to the exponent; and as multiplying by h affects not the exponents, but the co-efficients only; it follows that, in arranging the new product of $n+1$ binomials, according to the powers of x, we derive each of its terms from *two* terms of the previous product of n binomials, now made a multiplicand. Thus the co-efficient of the *third* term of the new product is $B+Ah$; that is, B the co-efficient of the *third* term of the multiplicand, $+$ the product of h into the co-efficient of the *second* term of the multiplicand. As B is the sum of the combinations of n letters taken two and two, and Ah is the sum of the same letters *multiplied* by h; the co-efficient $B+Ah$ is the sum of the combinations of $n+1$ letters, taken two and two.

In a similar manner, the term $\left.\begin{matrix}R\\Nh\end{matrix}\right\} x^{n-r+1}$ in the new product of $n+1$ binomials, is derived from the two terms Rx^{n-r} and Nx^{n-r+1} in the previous product of n binomials.

$$\left.\begin{array}{ll}\text{For} & Rx^{n-r}\quad\times x=Rx^{n-r+1}\\ \text{And} & Nx^{n-r+1}\times h=Nhx^{n-r+1}\end{array}\right\}$$

As R is the sum of the combinations of n letters taken r and r,

And Nh is the sum of the combinations of these letters taken $r-1$ and $r-1$, and *multiplied* by h; the co-efficient of x in the term $\left.\begin{matrix}R\\Nh\end{matrix}\right\} x^{n-r+1}$ *is the sum of the combinations of* $n+1$ *letters taken* r *and* r.

As m may represent, at pleasure, *any* term of the product of any number of binomial factors, the above law of the co-efficients may be considered universal.

493. To make the application to the *powers* of binomials, let the second terms of the factors $x+a$, $x+b$, $x+c$, &c. be supposed *equal*. Then will $ab=a^2$, $abc=a^3$, $abcd=a^4$, &c. And in the product of four binomial factors, (Art. 491,)

$$\left.\begin{matrix}a\\b\\c\\d\end{matrix}\right\}x^3=4ax^3.\qquad \left.\begin{matrix}ab\\ac\\bc\\ad\\bd\\cd\end{matrix}\right\}x^2=6a^2x^2.\qquad \left.\begin{matrix}abc\\abd\\acd\\bcd\end{matrix}\right\}x=4a^3x.\quad \text{That is,}$$

The co-efficient of $x^3=a$ repeated as many times, as there are letters *abcd.*

The co-efficient of $x^2=a^2$ repeated as often as there are combinations of these letters taken *two and two.*

The co-efficient of $x=a^3$ repeated as often as there are combinations of the letters taken *three and three.*

And if n be the index of *any* power to which $x+a$ is raised,

The co-efficient of $x^{n-1}=na$.

The co-efficient of $x^{n-2}=a^2$ repeated as many times, as there are combinations of n letters taken *two and two;* that is, $\frac{n}{1}\times\frac{n-1}{2}a^2$. (Art. 483.)

The co-efficient of $x^{n-3}=a^3$ repeated as often, as there are combinations of n letters taken three and three; that is,

$$\frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}a^3.$$

And as Rx^{n-r} is the term which has r terms before it, (Art. 491,) the co-efficient of x^{n-r} is a^r repeated as often, as there are combinations of n letters taken r and r; that is,

$$R=\frac{P}{Q}\times\frac{n-r+1}{r}a^r.$$

As a, in this formula, is the following quantity of the binomial $x+a$, it may be taken separately from the co-efficient of x, leaving $R=\frac{P}{Q}\times\frac{n-r+1}{r}$.

As r may be *any* number less than n, this law may be considered as applicable to any term, except the first, of the power of a binomial.

The preceding formula may be resolved into three factors,

$$\frac{P}{Q},\ n-r+1,\ \text{and}\ \frac{1}{r}.$$

As $\frac{P}{Q}$ is the number of combinations of n letters taken $r-1$ and $r-1$, it must be equal to *the co-efficient of the preceding term.* (Arts. 483, 491.)

As $n-r$ is the exponent of x in this mth term, and it decreases regularly by 1, $n-r+1$ must be equal to the exponent of x in the *preceding* term.

As r is the exponent of a in this term, and it increases regularly by 1, $r-1$ must be the exponent of a in the preceding term. Therefore the co-efficient of any term is equal to the co-efficient of the *preceding* term, multiplied by the exponent of the *leading* quantity in that term, and divided by the exponent of the *following* quantity increased by 1.

As the exponent of the following quantity in the second term is 1, and increases regularly by 1, it is evident that in each term, *it is equal to the number of the preceding terms.*

If, as in Art. 492, $(x+a)^n=x^n+Ax^{n-1}+Bx^{n-2}+Cx^{n-3}$, &c.
Then $A=n$, the co-efficient of the *second* term,

$$B=n\times\frac{n-1}{2},\quad \text{of the } \textit{third} \text{ term,}$$

$$C=n\times\frac{n-1}{2}\times\frac{n-2}{3},\quad \text{of the } \textit{fourth} \text{ term,}$$

$$D=n\times\frac{n-1}{2}\times\frac{n-2}{3}\times\frac{n-3}{4},\quad \text{of the } \textit{fifth} \text{ term, \&c.}$$

By the method in Art. 488, we obtain the co-efficients for the several terms in the power of a binomial, from the next *lower power.* By the propositions just demonstrated, we obtain the co-efficients for any term of a power of a binomial from the preceding *term.* Connecting these with the law of the *exponents,* as given in Art. 486, we have the following

Binomial Theorem.

494. *The* INDEX *of the leading quantity of the power of a binomial, begins in the first term with the index of the power, and decreases regularly by* 1. *The index of the following quantity begins with* 1 *in the second term, and increases regularly by* 1.

The CO-EFFICIENT *of the first term is* 1; *that of the second is equal to the index of the power; and universally, if the co-efficient of any term be multiplied by the index of the leading quantity in that term, and divided by the index of the following quantity increased by* 1, *it will give the co-efficient of the succeeding term.*

In algebraic characters, the theorem is

$$(a+b)^n=a^n+n\times a^{n-1}b+n\times\frac{n-1}{2}a^{n-2}b^2, \text{ \&c.}$$

It is here supposed, that the *terms* of the binomial have no other co-efficients or exponents than 1. Other binomials may be reduced to this form by substitution.

Ex. 1. What is the 6th power of $x+y$?

The terms without the co-efficients, are

$$x^6,\ x^5y,\ x^4y^2,\ x^3y^3,\ x^2y^4,\ xy^5,\ y^6.$$

And the co-efficients, are

$$1,\ 6,\ \frac{6\times5}{2},\ \frac{15\times4}{3},\ \frac{20\times3}{4},\ 6,\ 1.$$

that is, $\quad 1,\ 6,\ 15,\ \ 20,\ \ 15,\ \ 6,\ 1.$

Prefixing these to the several terms, we have the power required;

$$x^6+6x^5y+15x^4y^2+20x^3y^3+15x^2y^4+6xy^5+y^6.$$

2. $(d+h)^5=d^5+5d^4h+10d^3h^2+10d^2h^3+5dh^4+h^5.$

3. What is the nth power of $b+y$?

Ans. $b^n+Ab^{n-1}y+Bb^{n-2}y^2+Cb^{n-3}y^3+Db^{n-4}y^4$, &c.

That is, supplying the co-efficients which are here represented by A, B, C, &c.

$$b^n+n\times b^{n-1}y+n\times\frac{n-1}{2}\times b^{n-2}y^2,\ \&c.$$

4. What is the 5th power of x^2+3y^2?

Substituting a for x^2, and b for $3y^2$, we have

$$(a+b)^5=a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5,$$

And restoring the values of a and b,

$(x^2+3y^2)^5=x^{10}+15x^8y^2+90x^6y^4+270x^4y^6+405x^2y^8+243y^{10}.$

5. What is the sixth power of $(3x+2y)$?

Ans. $729x^6+2916x^5y+4860x^4y^2+4320x^3y^3+2160x^2y^4+576xy^5+64y^6.$

495. A *residual* quantity may be involved in the same manner, without any variation except in the *signs*. By repeated multiplications, as in Art. 230, we obtain the following powers of $(a-b)$.

$$(a-b)^2=a^2-2ab+b^2.$$
$$(a-b)^3=a^3-3a^2b+3ab^2-b^3.$$
$$(a-b)^4=a^4-4a^3b+6a^2b^2-4ab^3+b^4,\ \&c.$$

By comparing these with the like powers of $(a+b)$ in Art. 485, it will be seen, that there is no difference except in the *signs*. There, *all* the terms are positive. Here, the terms which contain the *odd* powers of b are negative. See Art. 233.

The sixth power of $(x-y)$ is

$$x^6-6x^5y+15x^4y^2-20x^3y^3+15x^2y^4-6xy^5+y^6.$$

The nth power of $(a-b)$ is

$$a^n-Aa^{n-1}b+Ba^{n-2}b^2-Ca^{n-3}b^3, \text{ \&c.}$$

496. When one of the terms of a binomial is a *unit*, it is generally omitted in the power, except in the first or last term; because every power of 1 is 1, (Art. 226,) and this when it is a factor, has no effect upon the quantity with which it is connected. (Art. 85.)

Thus the cube of $(x+1)$ is $\quad x^3+3x^2\times1+3x\times1^2+1^3$,
Which is the same as $\quad x^3+3x^2+3x+1$.

The insertion of the powers of 1 is of no use, unless it be to preserve the *exponents* of both the leading and the following quantity in each term, for the purpose of finding the coefficients. But this will be unnecessary, if we bear in mind, that the *sum* of the two exponents, in each term, is equal to the index of the power. (Art. 485.) So that, if we have the exponent of the *leading* quantity, we may know that of the *following* quantity, and *v. v.*

Ex. 1. The sixth power of $(1-y)$ is

$$1-6y+15y^2-20y^3+15y^4-6y^5+y^6.$$

2. $(1+x)^n=1+Ax+Bx^2+Cx^3+Dx^4$, &c.

497. From the comparatively simple manner in which the power is expressed, when the first term of the root is a unit, is suggested the expediency of reducing other binomials to this form.

The quotient of $(a+x)$ divided by a is $\left(1+\frac{x}{a}\right)$. This multiplied into the divisor, is equal to the dividend; that is, $(a+x)=a\times\left(1+\frac{x}{a}\right)$; therefore $(a+x)^n=a^n\times\left(1+\frac{x}{a}\right)^n$.

By expanding the factor $\left(1+\frac{x}{a}\right)^n$, we have

$$(a+x)^n=a^n\times\left(1+\frac{x}{a}\right)^n=a^n\times\left(1+A\frac{x}{a}+B\frac{x^2}{a^2}+C\frac{x^3}{a^3}\right) \text{ \&c.}$$

498. Let $A+Bx+Cx^2+Dx^3+$ &c.$=M+Nx+Px^2+Qx^3+$ &c. or (Art. 183,)

$$(A-M)+(B-N)x+(C-P)x^2+(D-Q)x^3+\text{ &c. }=0.$$

An equation of this form is said to be *identical*, if the equality subsists, *whatever value* be given to x. (Art. 173.) In such an equation, the *co-efficients* of the several powers of x, as well as the aggregate of the terms into which x does *not* enter, must each $=0$. That is,

$$A-M=0,\ B-N=0,\ C-P=0,\text{ &c.}$$

For if the equation holds true, whatever be the value of x, it must do so, when x is an *infinitesimal.* But on this supposition,

$(B-N)x+(C-P)x^2+(D-Q)x^3+$ &c. each $=0$. (Art. 469.)

Unless, then, in the equation,

$$(A-M)+(B-N)x+(C-P)x^2+(D-Q)x^3+\text{ &c. }=0,$$

the remaining term $(A-M)=0$, the whole of this member of the equation can not $=0$.

Dividing $(B-N)x+(C-P)x^2+(D-Q)x^3+$ &c. by x,

We have $(B-N)\ +(C-P)x+(D-Q)x^2+$ &c. $=0$.

Therefore $(B-N)=0$. Dividing again by x, we have $(C-P)=0$.

Hence $A=M$, $B=N$, $C=P$, $D=Q$, &c.

499. In the demonstration which has been given of the binomial theorem (Arts. 491, 2, 3,) the exponent of the binomial has been supposed to be a *positive whole number.* But the theorem is applicable to cases in which the exponent is *fractional,* and either *positive* or *negative.* To prove this,

Let $(1+x)^{\frac{m}{n}}=1+Ax+Bx^2+Cx^3+$ &c.

And $(1+y)^{\frac{m}{n}}=1+Ay+By^2+Cy^3+$ &c.

A, B, C, &c. being the co-efficients whose values are to be determined.

Subtracting one of these series from the other, we have

$$(1+x)^{\frac{m}{n}}-(1+y)^{\frac{m}{n}}=A.(x-y)+B.(x^2-y^2)+C.(x^3-y^3)+\text{&c.}$$

Put $(1+x)=z^n$, and $(1+y)=v^n$. Then $z^n-v^n=x-y$.

Applying the exponent $\frac{m}{n}$ to both members of the equations $(1+x)=z^n$, and $(1+y)=v^n$, (Art. 234,) we have

$$(1+x)^{\frac{m}{n}}=z^m,\text{ and }(1+y)^{\frac{m}{n}}=v^m.\quad\text{Therefore,}$$

$$z^m-v^m=A.(x-y)+B.(x^2-y^2)+C.(x^3-y^3)+\text{ &c.}$$

And as $z^n-v^n=x-y$,

$$\frac{z^m-v^m}{z^n-v^n}=A+B.\frac{x^2-y^2}{x-y}+C.\frac{x^3-y^3}{x-y}+\text{ \&c. which is equal to}$$

$$A+B.(x+y)+C.(x^2+xy+y^2)+\text{ \&c. (Art. 130.)}$$

Dividing the numerator and denominator of the fraction $\frac{z^m-v^m}{z^n-v^n}$ by $z-v$, (Art. 130,) we have

$$\frac{z^{m-1}+z^{m-2}v+z^{m-3}v^2\ldots v^{m-1}}{z^{n-1}+z^{n-2}v+z^{n-3}v^2\ldots v^{n-1}}=A+B.(x+y)+C.(x^2+xy+y^2)+\text{\&c.}$$

As x and y may be of any value, let $x=y$.
Then $z=v$, each term in the numerator becomes z^{m-1}, and each in the denominator z^{n-1}. Observing then, that the *number* of terms in one $=m$, and in the other $=n$, (Art. 130,) we have

$$\frac{mz^{m-1}}{nz^{n-1}}=\frac{mz^m}{nz^n}=A+2Bx+3Cx^2+\text{ \&c.}$$

Multiplying the first member of the equation by z^n, and the second by $1+x$ which $=z^n$; and observing that $z^m=(1+x)^{\frac{m}{n}}$,

we have $$\left.\begin{aligned}\frac{m}{n}.(1+x)^{\frac{m}{n}}=A+2Bx+3Cx^2&+\text{ \&c.}\\ +\ Ax+2Bx^2&+\text{ \&c.}\end{aligned}\right\}$$

But multiplying by $\frac{m}{n}$ the equation at the beginning of this article, we have

$$\frac{m}{n}.(1+x)^{\frac{m}{n}}=\frac{m}{n}+\frac{m}{n}Ax+\frac{m}{n}Bx^2+\frac{m}{n}Cx^3+\text{ \&c.}$$

As the members of these two equations are equal, (Ax. 11,) and as the powers of x, in the corresponding terms are the same in both, the *co-efficients*, which are independent of any particular values of x, must be equal, the first term of one being equal to the first term of the other, the second of one, to the second of the other, &c. (Art. 498.) By comparing them, we have

$\frac{m}{n}=A$ Hence $A=\frac{m}{n}$

$\frac{m}{n}A=2B+A$ $B=A.\left(\frac{\frac{m}{n}-1}{2}\right)=\frac{m}{n}\left(\frac{\frac{m}{n}-1}{2}\right)$

$\frac{m}{n}B=3C+2B \qquad C=B.\left(\frac{\frac{m}{n}-2}{3}\right)=\frac{m}{n}\left(\frac{\frac{m}{n}-1}{2}\right)\left(\frac{\frac{m}{n}-2}{3}\right)$

This result agrees with the rule for the co-efficients, in the binomial theorem. (Art. 494.)

If n be put $=1$, the exponent $\frac{m}{n}=m$, a positive *whole number*.

And $A=m$, the co-efficient of the *second* term.

$B=m\left(\frac{m-1}{2}\right)$ of the *third* term.

$C=m\left(\frac{m-1}{2}\right)\left(\frac{m-2}{3}\right)$ of the *fourth* term, as in Art. 493.

500. The demonstration is nearly the same, when the exponent of the binomial $1+x$ is *negative*, $-\frac{m}{n}$ being substituted for $\frac{m}{n}$, and due regard being paid, at every step, to the *sign* of the exponent.

Let $(1+x)^{-\frac{m}{n}}=1+Ax+Bx^2+Cx^3+$ &c.

And $(1+y)^{-\frac{m}{n}}=1+Ay+By^2+Cy^3+$ &c.

By subtraction,

$(1+x)^{-\frac{m}{n}}-(1+y)^{-\frac{m}{n}}=A.(x-y)+B.(x^2-y^2+C).(x^3-y^3)+$ &c.

Put $(1+x)=z^n$, and $(1+y)=v^n$. Then $z^n-v^n=x-y$.

Applying the exponent $-\frac{m}{n}$, as $+\frac{m}{n}$ was, in Art. 499, we have

$(1-x)^{-\frac{m}{n}}=z^{-m}$, and $(1+y)^{-\frac{m}{n}}=v^{-m}$. Therefore,

$z^{-m}-v^{-m}=A.(x-y)+B.(x^2-y^2)+C.(x^3-y^3)+$ &c.

And as $z^n-v^n=x-y$,

$\frac{z^{-m}-v^{-m}}{z^n-v^n}=A+B.(x+y)+C.(x^2+xy+y^2)+$ &c. (Art. 130.)

The numerator $z^{-m}-v^{-m}=\frac{1}{z^m}-\frac{1}{v^m}$ (Art. 224) $=\frac{v^m-z^m}{z^mv^m}$ (Art. 150.)

Hence $\frac{z^{-m}-v^{-m}}{z^n-v^n}=\frac{1}{z^mv^m}\left(\frac{v^m-z^m}{z^n-v^n}\right)=-\frac{1}{z^mv^m}\left(\frac{z^m-v^m}{z^n-v^n}\right)$, (Art. 148.)

It has been shown, (Art. 499,) that when $z=v$,

$$\frac{z^m-v^m}{z^n-v^n} \text{ becomes } \frac{mz^{m-1}}{nz^{n-1}}, \text{ or } \frac{mz^m}{nz^n}.$$

On the same supposition,

$$\frac{z^{-m}-v^{-m}}{z^n-v^n}=-\frac{1}{z^{2m}}\left(\frac{mz^{m-1}}{nz^{n-1}}\right)=-\frac{mz^{-m-1}}{nz^{n-1}}=-\frac{mz^{-m}}{nz^n}.$$

Therefore,

$$\frac{z^{-m}-v^{-m}}{z^n-v^n} \text{ or } \frac{-mz^{-m}}{nz^n}=A+B.(x+y)+C.(x^2+xy+y^2)+\&c.$$

Multiplying the first member of the equation by z^n, and the second by $1+x$ which $=z^n$, and observing that $z^{-m}=(1+x)^{-\frac{m}{n}}$ we have

$$-\frac{m}{n}\cdot(1+x)^{-\frac{m}{n}}=\left.\begin{array}{l}A+2Bx+3Cx^2+\&c.\\ \quad+\ Ax+2Bx^2+\&c.\end{array}\right\}$$

Multiplying by $-\frac{m}{n}$ the equation at the beginning of this article,

$$-\frac{m}{n}\cdot(1+x)^{-\frac{m}{n}}=-\frac{m}{n}-\frac{m}{n}Ax-\frac{m}{n}Bx^2-\frac{m}{n}Cx^2-\&c.$$

Comparing the two equations, we have

$-\frac{m}{n}=A$ Therefore $A=-\frac{m}{n}$

$-\frac{m}{n}A=2B+A \qquad B=A.\left(\frac{-\frac{m}{n}-1}{2}\right)=-\frac{m}{n}\left(\frac{-\frac{m}{n}-1}{2}\right)$

$-\frac{m}{n}B=3C+2B \qquad C=B.\left(\frac{-\frac{m}{n}-2}{3}\right)=-\frac{m}{n}\left(\frac{-\frac{m}{n}-1}{2}\right)\left(\frac{-\frac{m}{n}-2}{3}\right)$

The co-efficient of the second term is *negative.*

That of the third term, being the product of *two* negative factors is *positive.*

That of the fourth term, the product of *three* negatives is *negative.* That is, when a binomial whose exponent is a negative fraction is expanded, the co-efficients of the several terms are *alternately positive and negative.*

501. The exponent $-\frac{m}{n}$, when $n=1$, becomes $-m$, *a negative whole number;* and

$A=-m$, the co-efficient of the *second* term.

$B=-m\left(\frac{-m-1}{2}\right)$ of the *third* term.

$C=-m\left(\frac{-m-1}{2}\right)\left(\frac{-m-2}{3}\right)$ of the *fourth* term.

The same as in Art. 493, except that the co-efficients here are *alternately positive and negative.*

We have then, in this and preceding articles, demonstrations of the binomial theorem, for each of the *four cases,* in which the exponent of the binomial is either *a positive whole number, a positive fraction, a negative fraction, or a negative whole number.*

502. When the index of the power to which any binomial is to be raised is a *positive whole number,* the series will *terminate.* The number of terms will be limited, as in all the preceding examples.

For, as the index of the leading quantity continually decreases by one, it must, in the end, become 0, and then the series will break off.

Thus the 5th term of the fourth power of $a+x$ is x^4, or a^0x^4, a^0 being commonly omitted, because it is equal to 1. (Art. 224.) If we attempt to continue the series farther, the co-efficient of the next term, according to the rule, will be $\frac{1\times0}{5}=0$. (Art. 106.) And as the co-efficients of all succeeding terms must depend on this, they will also be 0.

503. If the index of the proposed power is *negative,* this can never become 0, by the successive subtractions of a unit. The series will, therefore, *never terminate;* but like many decimal fractions, may be continued to any extent that is desired.

Ex. Expand into a series $\frac{1}{(a+y)^2}=(a+y)^{-2}$.

The terms without the co-efficients, are

$$a^{-2},\ a^{-3}y,\ a^{-4}y^2,\ a^{-5}y^3,\ a^{-6}y^4, \text{ \&c.}$$

The co-efficient of the 2d term is -2, of the 4th $\frac{+3\times-4}{3}=-4$.

Of the third, $\frac{-2\times-3}{2}=+3$, of the 5th, $\frac{-4\times-5}{4}=+5$.

The series then is

$$a^{-2}-2a^{-3}y+3a^{-4}y^2-4a^{-5}y^3+5a^{-6}y^4, \text{ \&c.}$$

Here the law of the progression is apparent; the co-efficients increase regularly by 1, and their signs are alternately positive and negative.

504. The Binomial Theorem is of great utility, not only in raising powers, but particularly in finding the *roots* of binomials. A root may be expressed in the same manner as a power, except that the exponent is, in the one case an *integer*, in the other a *fraction*. (Art. 256.) Thus $(a+b)^n$ may be either a power or a root. It is a power if $n=2$, but a root if $n=\frac12$.

505. If a root be expanded by the binomial theorem, the series *will never terminate*. A series produced in this way terminates, only when the index of the leading quantity becomes equal to 0, so as to destroy the co-efficients of the succeeding terms. (Art. 502.) But according to the theorem, the difference in the index, between one term and the next, is always a unit; and a *fraction*, though it may change from positive to negative, can not become exactly equal to 0, by successive subtractions of units. Thus, if the index in the first term be $\frac12$, it will be,

In the 2d, $\frac12-1=-\frac12$, In the 4th, $-\frac32-1=-\frac52$,
In the 3d, $-\frac12-1=-\frac32$, In the 5th, $-\frac52-1=-\frac72$, &c.

Ex. What is the square root of $(a+b)$?

The terms, without the co-efficients, are,

$$a^{\frac12},\ a^{-\frac12}b,\ a^{-\frac32}b^2,\ a^{-\frac52}b^3,\ a^{-\frac72}b^4, \text{ \&c.}$$

The co-efficient of the second term is $+\frac12$,

of the 3d, $\frac{\frac12\times-\frac12}{2}=-\frac18$, of the 4th, $\frac{-\frac18\times-\frac32}{3}=+\frac1{16}$.

And the series is $a^{\frac12}+\frac12a^{-\frac12}b-\frac18a^{-\frac32}b^2+\frac1{16}a^{-\frac52}b^3$, &c.

When a quantity is expanded by the Binomial Theorem, the law of the series will frequently be more apparent, if the *factors*, by which the co-efficients are formed, *are kept distinct.*

1. Expand into a series $(a^2+x)^{\frac12}$.

Substituting b for a^2, we have

$$(b+x)^{\frac{1}{2}}=b^{\frac{1}{2}}+Ab^{-\frac{1}{2}}x+Bb^{-\frac{3}{2}}x^2+Cb^{-\frac{5}{2}}x^3+Db^{-\frac{7}{2}}x^4, \text{ \&c.}$$

$$A=\frac{1}{2}.$$

$$B=\frac{1}{2}\times\frac{-\frac{1}{2}}{2}=\frac{1}{2}\times-\frac{1}{4}=-\frac{1}{2.4}.$$

$$C=-\frac{1}{2.4}\times\frac{-\frac{3}{2}}{3}=-\frac{1}{2.4}\times-\frac{3}{6}=\frac{3}{2.4.6}.$$

$$D=\frac{3}{2.4.6}\times\frac{-\frac{5}{2}}{4}=\frac{3}{2.4.6}\times-\frac{5}{8}=-\frac{3.5}{2.4.6.8}.$$

Restoring, then, the value of b, and writing $\frac{1}{a}$ for a^{-1}, we have

$$(a^2+x)^{\frac{1}{2}}=a+\frac{x}{2a}-\frac{x^2}{2.4a^3}+\frac{3x^3}{2.4.6a^5}-\frac{3.5x^4}{2.4.6.8a^7}, \text{ \&c.}$$

2. Expand into a series $(1+x)^{\frac{1}{2}}$.

Ans. $1+\frac{x}{2}-\frac{x^2}{2.4}+\frac{3x^3}{2.4.6}-\frac{3.5x^4}{2.4.6.8}$, &c.

3. Expand $\sqrt{2}$, or $(1+1)^{\frac{1}{2}}$.

Ans. $1+\frac{1}{2}-\frac{1}{2.4}+\frac{3}{2.4.6}-\frac{3.5}{2.4.6.8}+\frac{3.5.7}{2.4.6.8.10}$, &c.

4. Expand $(a+x)^{\frac{1}{2}}$, or $a^{\frac{1}{2}}\times\left(1+\frac{x}{a}\right)^{\frac{1}{2}}$. See Art. 497.

Ans. $a^{\frac{1}{2}}\times\left(1+\frac{x}{2a}-\frac{x^2}{2.4a^2}+\frac{3x^3}{2.4.6a^3}-\frac{3.5x^4}{2.4.6.8a^4}, \text{ \&c.}\right)$

5. Expand $(a+b)^{\frac{1}{3}}$, or $a^{\frac{1}{3}}\times\left(1+\frac{b}{a}\right)^{\frac{1}{3}}$.

Ans. $a^{\frac{1}{3}}\times\left(1+\frac{b}{3a}-\frac{2b^2}{3.6a^2}+\frac{2.5b^3}{3.6.9a^3}-\frac{2.5.8b^4}{3.6.9.12a^4}, \text{ \&c.}\right)$

6. Expand into a series $(a-b)^{\frac{1}{4}}$.

Ans. $a^{\frac{1}{4}}\times\left(1-\frac{b}{4a}-\frac{3b^2}{4.8a^2}-\frac{3.7b^3}{4.8.12a^3}-\frac{3.7.11b^4}{4.8.12.16a^4}, \text{\&c.}\right)$

7. Expand $(a+x)^{-\frac{1}{2}}$.

8. Expand $(1-x)^{\frac{2}{3}}$.

9. Expand $(1+x)^{-\frac{1}{5}}$.

10. Expand $(a^2+x)^{-\frac{1}{2}}$.

506. The binomial theorem may also be applied to quantities consisting of *more than two terms.* By substitution, several terms may be reduced to two, and when the compound expressions are restored, such of them as have exponents may be separately expanded.

Ex. What is the cube of $a+b+c$?

Substituting h for $(b+c)$, we have $a+(b+c)=a+h$.

And by the theorem, $(a+h)^3=a^3+3a^2h+3ah^2+h^3$.

That is, restoring the value of h,

$$(a+b+c)^3=a^3+3a^2\times(b+c)+3a\times(b+c)^2+(b+c)^3.$$

The two last terms contain powers of $(b+c)$; but these may be separately involved.

Promiscuous Examples.

1. What is the 8th power of $(a+b)$?

Ans. $a^8+8a^7b+28a^6b^2+56a^5b^3+70a^4b^4+56a^3b^5+28a^2b^6+8ab^7+b^8$.

2. What is the 7th power of $(a-b)$?

3. Expand into a series $\frac{1}{1-a}$, or $(1-a)^{-1}$.

Ans. $1+a+a^2+a^3+a^4+a^5$, &c.

4. Expand $\frac{h}{a-b}$, or $h\times(a-b)^{-1}$.

Ans. $h\times\left(\frac{1}{a}+\frac{b}{a^2}+\frac{b^2}{a^3}+\frac{b^3}{a^4}, \text{\&c.}\right)$ or $\frac{h}{a}+\frac{bh}{a^2}+\frac{b^2h}{a^3}+\frac{b^3h}{a^4}$, &c

5. Expand into a series $(a^2+b^2)^{\frac{1}{2}}$.

Ans. $a+\frac{b^2}{2a}-\frac{b^4}{8a^3}+\frac{b^6}{16a^5}$, &c.

6. Expand into a series $(a+y)^{-4}$.

Ans. $\frac{1}{a^4}-\frac{4y}{a^5}+\frac{10y^2}{a^6}-\frac{20y^3}{a^7}+\frac{35y^4}{a^8}$, &c.

7. Expand into a series $(c^3+x^3)^{\frac{1}{3}}$.

Ans. $c\times\left(1+\frac{x^3}{3c^3}-\frac{2x^6}{3.6c^6}+\frac{2.5x^9}{3.6.9c^9}, \text{\&c.}\right)$

8. Expand $\dfrac{d}{\sqrt{c^2+x^2}}$ or $d(c^2+x^2)^{-\frac{1}{2}}$.

Ans. $\dfrac{d}{c}\left(1-\dfrac{x^2}{2c^2}+\dfrac{3x^4}{2.4c^4}-\dfrac{3.5x^6}{2.4.6c^6}+\dfrac{3.5.7x^8}{2.4.6.8c^8}, \text{\&c.}\right)$

9. Find the 5th power of (a^2+y^3).

10. Find the 4th power of $(a+b+x)$.

11. Expand $(a^3-x)^{\frac{1}{2}}$.

12. Expand $(1-y^2)^{\frac{1}{2}}$.

13. Expand $(a-x)^{\frac{1}{3}}$.

14. Expand $h(a^3-y^3)^{\frac{1}{3}}$.

SECTION XVII.

EVOLUTION OF COMPOUND QUANTITIES.

ART. **507.** THE roots of compound quantities may be extracted by the following general rule:

After arranging the terms according to the powers of one of the letters, so that the highest power shall stand first, the next highest next, &c.

Take the root of the first term, for the first term of the required root:

Subtract the power from the given quantity, and divide the first term of the remainder, by the first term of the root involved to the next inferior power, and multiplied by the index of the given power; the quotient will be the next term of the root.*

Subtract the power of the terms already found from the given quantity, and using the same divisor, proceed as before.

This rule verifies itself. For the root, whenever a new term is added to it, is involved, for the purpose of subtracting

* By the *given power* is meant a power of the same name with the required root. As powers and roots are correlative, any quantity is the square of its square root, the cube of its cube root, &c.

its power from the given quantity: and when the power is *equal* to this quantity, it is evident the true root is found.

Ex. 1. Extract the cube root of

$$a^6+3a^5-3a^4-11a^3+6a^2+12a-8(a^2+a-2.$$

a^6, the first subtrahend.

$3a^4)*\ \ 3a^5$, &c. the first remainder.

$a^6+3a^5+3a^4+a^3$, the second subtrahend.

$3a^4)*\ \ *\ \ -6a^4$, &c. the second remainder.

$$a^6+3a^5-3a^4-11a^3+6a^2+12a-8.$$

Here a^2, the cube root of a^6, is taken for the first term of the required root. The power a^6 is subtracted from the given quantity. For a divisor, the first term of the root is squared, that is, raised to the next inferior power, and multiplied by 3, the index of the given power.

By this, the first term of the remainder $3a^5$, &c. is divided, and the quotient a is added to the root. Then a^2+a, the part of the root now found, is involved to the cube, for the second subtrahend, which is subtracted from the whole of the given quantity. The first term of the remainder $-6a^4$, &c. is divided by the divisor used above, and the quotient -2 is added to the root. Lastly the whole root is involved to the cube, and the power is found to be exactly equal to the given quantity.

It is not necessary to write the remainder at length, as, in dividing, the first term only is wanted.

2. Extract the fourth root of

$$a^4+8a^3+24a^2+32a+16(a+2$$

a^4

$4a^3)*\ \ 8a^3$, &c.

$$a^4+8a^3+24a^2+32a+16.$$

3. What is the fifth root of

$a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5$? Ans. $a+b$.

4. What is the cube root of

$a^3-6a^2b+12ab^2-8b^3$? Ans. $a-2b$.

5. What is the square root of

$$
\begin{array}{ll}
 & 4a^2-12ab+9b^2+16ah-24bh+16h^2(2a-3b+4h \\
 & 4a^2 \\
\hline
4a) & *-12ab, \text{ \&c.} \\
\hline
 & 4a^2-12ab+9b^2 \\
\hline
4a) & * \quad * \quad *+ \quad 16ah, \text{ \&c.} \\
\hline
 & 4a^2-12ab+9b^2+16ah-24bh+16h^2.
\end{array}
$$

In finding the divisor here, the term $2a$ in the root is not involved, because the power next below the square is the first power.

508. But the *square* root is more commonly extracted by the following rule, which is of the same nature as that which is used in Arithmetic.

After arranging the terms according to the powers of one of the letters, take the root of the first term, for the first term of the required root, and subtract the power from the given quantity.

Bring down two other terms for a dividend. Divide by double the root already found, and add the quotient, both to the root, and to the divisor. Multiply the divisor thus increased, into the term last placed in the root, and subtract the product from the dividend.

Bring down two or three additional terms and proceed as before.

Ex. 1. What is the square root of

$$
\begin{array}{lll}
 & a^2+2ab+b^2+2ac+2bc+c^2(a+b+c. & \\
 & a^2, & \text{the first subtrahend.} \\
\hline
2a+b) & * \quad 2ab+b^2 & \\
\text{Into } b= & 2ab+b^2, & \text{the second subtrahend.} \\
\hline
2a+2b+c) & * \quad * \quad 2ac+2bc+c^2 & \\
\text{Into } c= & 2ac+2bc+c^2, & \text{the third subtrahend.}
\end{array}
$$

Here it will be seen, that the several subtrahends are successively taken from the given quantity, till it is exhausted. If then, these subtrahends are together equal to the square of the terms placed in the root, the root is truly assigned by the rule.

The *first* subtrahend is the square of the first term of the root.

The *second* subtrahend is the product of the second term of the root, into itself, and into twice the preceding term.

The *third* subtrahend is the product of the third term of the root, into itself, and into twice the sum of the two preceding terms, &c.

That is, the subtrahends are equal to

$$a^2+(2a+b)\times b+(2a+2b+c)\times c, \text{ \&c.}$$

and this expression is equal to the square of the root.

For $(a+b)^2=a^2+2ab+b^2=a^2+(2a+b)\times b$. (Art. 119.)

And putting $h=a+b$, the square $h^2=a^2+(2a+b)\times b$.

And $(a+b+c)^2=(h+c)^2=h^2+(2h+c)\times c$;

that is, restoring the values of h and h^2,

$$(a+b+c)^2=a^2+(2a+b)\times b+(2a+2b+c)\times c.$$

In the same manner, it may be proved, that, if another term be added to the root, the power will be increased, by the product of that term into itself, and into twice the sum of the preceding terms.

The demonstration will be substantially the same, if some of the terms be *negative*.

2. What is the square root of

$$\begin{array}{lll}
 & 1-4b+4b^2+2y-4by+y^2\,(1-2b+y & \\
 & 1 & \\
\hline
2-2b) & *-4b+4b^2 & \\
\text{Into } -2b= & -4b+4b^2 & \\
\hline
2-4b+y) & *\quad * & 2y-4by+y^2 \\
\text{Into } \quad y= & & 2y-4by+y^2
\end{array}$$

3. What is the square root of

$a^6-2a^5+3a^4-2a^3+a^2$? Ans. a^3-a^2+a.

4. What is the square root of

$a^4+4a^2b+4b^2-4a^2-8b+4$? Ans. a^2+2b-2.

It will frequently facilitate the extraction of roots, to consider the index as composed of two or more *factors*.

Thus $a^{\frac{1}{4}}=a^{\frac{1}{2}\times\frac{1}{2}}$. (Art. 269.) And $a^{\frac{1}{6}}=a^{\frac{1}{3}\times\frac{1}{2}}$. That is,

The fourth root is equal to the square root of the square root;

The sixth root is equal to the square root of the cube root;

The eighth root is equal to the square root of the fourth root, &c.

To find the sixth root, therefore, we may first extract the cube root, and then the square root of this.

1. Find the square root of $x^4-4x^3+6x^2-4x+1$.
2. Find the cube root of $x^6-6x^5+15x^4-20x^3+15x^2-6x+1$.
3. Find the square root of $4x^4-4x^3+13x^2-6x+9$.
4. Find the 4th root of $16a^4-96a^3x+216a^2x^2-216ax^3+81x^4$.
5. Find the 5th root of $x^5+5x^4+10x^3+10x^2+5x+1$.
6. Find the 6th root of

$$a^6-6a^5b+15a^4b^2-20a^3b^3+15a^2b^4-6ab^5+b^6.$$

Roots of Binomial Surds.

509. It is sometimes expedient to express the square root of a quantity of the form $a\pm\sqrt{b}$, called a binomial or residual surd, by the sum or difference of two other surds. A formula for this purpose may be derived from the following propositions:

1. The square root of a whole number can not consist of *two parts*, one of which is *rational*, and the other *a surd.*

If it be possible, let $\sqrt{a}=x+\sqrt{y}$, in which the part x is rational.

Squaring both sides, $a=x^2+2x\sqrt{y}+y$

And reducing, $\sqrt{y}=\dfrac{a-x^2-y}{2x}$, a rational quantity; which is contrary to the supposition.

2. In every equation of the form $x+\sqrt{y}=a+\sqrt{b}$, the rational parts on each side are *equal*, and also the remaining parts.

If x be not equal to a, let $x=a\pm z$.

Then $a\pm z+\sqrt{y}=a+\sqrt{b}$. And $\sqrt{b}=z+\sqrt{y}$;

That is, $\sqrt{b}$ consists of two parts, one of which is rational, and the other not; which, according to the preceding proposition, is impossible.

In the same manner it may be shewn, that in the equation, $x-\sqrt{y}=a-\sqrt{b}$, the rational parts on each side are equal, and also the remaining parts.

3. If $\sqrt{a+\sqrt{b}}=x+\sqrt{y}$, then $\sqrt{a-\sqrt{b}}=x-\sqrt{y}$.

For, by squaring the first equation, we have

$$a+\sqrt{b}=x^2+2x\sqrt{y}+y$$

And by the last proposition,

$$a=x^2+y$$
$$\sqrt{b}=2x\sqrt{y}$$

By subtraction, $\quad a-\sqrt{b}=x^2-2x\sqrt{y}+y$

By evolution, $\quad \sqrt{a-\sqrt{b}}=x-\sqrt{y}$

510. To find, now, an expression for the square root of a binomial or residual surd,

Let $\quad \sqrt{a+\sqrt{b}}=x+\sqrt{y}$

Then $\quad \sqrt{a-\sqrt{b}}=x-\sqrt{y}$

Squaring both sides of each, we have

$$a+\sqrt{b}=x^2+2x\sqrt{y}+y$$
$$a-\sqrt{b}=x^2-2x\sqrt{y}+y$$

Adding the two last, and dividing, $\quad a=x^2+y$

Multiplying the two first, $\quad \sqrt{a^2-b}=x^2-y$

Adding and subtracting,

$$a+\sqrt{a^2-b}=2x^2. \quad \text{Or} \quad x=\sqrt{\frac{a+\sqrt{a^2-b}}{2}}$$

$$a-\sqrt{a^2-b}=2y \quad \text{And} \quad \sqrt{y}=\sqrt{\frac{a-\sqrt{a^2-b}}{2}}$$

Therefore, as $\sqrt{a+\sqrt{b}}=x+\sqrt{y}$, and $\sqrt{a-\sqrt{b}}=x-\sqrt{y}$,

$$\sqrt{a+\sqrt{b}}=\sqrt{\frac{a+\sqrt{a^2-b}}{2}}+\sqrt{\frac{a-\sqrt{a^2-b}}{2}}$$

$$\sqrt{a-\sqrt{b}}=\sqrt{\frac{a+\sqrt{a^2-b}}{2}}-\sqrt{\frac{a-\sqrt{a^2-b}}{2}}$$

Or, substituting d for $\sqrt{a^2-b}$,

1. $\sqrt{a+\sqrt{b}}=\sqrt{\tfrac{1}{2}(a+d)}+\sqrt{\tfrac{1}{2}(a-d)}$.
2. $\sqrt{a-\sqrt{b}}=\sqrt{\tfrac{1}{2}(a+d)}-\sqrt{\tfrac{1}{2}(a-d)}$.

Ex. 1. Find the square root of $3+2\sqrt{2}$.

Here $a=3$, $a^2=9$, $\sqrt{b}=2\sqrt{2}$, $b=8$, $a^2-b=9-8=1$.

Therefore $\sqrt{3+2\sqrt{2}}=\sqrt{\frac{3+1}{2}}+\sqrt{\frac{3-1}{2}}=\sqrt{2}+1$.

2. Find the square root of $11+6\sqrt{2}$. $\quad$ Ans. $3+\sqrt{2}$.

3. Find the square root of $6-2\sqrt{5}$. Ans. $\sqrt{5}-1$.
4. Find the square root of $7+4\sqrt{3}$. Ans. $2+\sqrt{3}$.
5. Find the square root of $7-2\sqrt{10}$. Ans. $\sqrt{5}-\sqrt{2}$.

These results may be verified, in each instance, by multiplying the root into itself, and thus re-producing the binomial from which it is derived.

SECTION XVIII.

INFINITE SERIES.

ART. **511.** IT is frequently the case, that, in attempting to extract the root of a quantity, or to divide one quantity by another, we find it impossible to assign the quotient or root with exactness. But, by continuing the operation, one term after another may be added, so as to bring the result nearer and nearer to the value required. When the number of terms is supposed to be extended beyond any determinate limits the expression is called an *infinite series*. The *quantity*, however, may be finite, though the number of terms be unlimited.

An infinite series may appear, at first view, much less simple than the expression from which it is derived. But the former is, frequently, more within the power of calculation than the latter. Much of the labor and ingenuity of mathematicians has, accordingly, been employed on the subject of series. If it were necessary to find each of the terms by actual calculation, the undertaking would be hopeless. But a few of the leading terms will, generally, be sufficient to determine the law of the progression.

512. A *fraction* may often be expanded into an infinite series, *by dividing the numerator by the denominator*. For the *value* of a fraction is equal to the quotient of the numerator divided by the denominator. (Art. 140.) When this quotient can not be expressed, in a limited number of terms, it may be represented by an infinite series.

Ex. 1. To reduce the fraction $\frac{1}{1-a}$ to an infinite series divide 1 by $1-a$, according to the rule in Art. 126.

The quotient is $1+a+a^2+a^3+a^4+a^5+a^6$, &c. which shows that the series, after the first term, consists of the powers of a, rising regularly one above another.

That the series may *converge*, that is, come nearer and nearer to the exact value of the fraction, it is necessary that the first term of the divisor be greater than the second. In the example just given, 1 must be greater than a. For at each step of the division, there is a *remainder;* and the quotient is not complete, till this is placed over the divisor and annexed. Now the first remainder is a, the second a^2, the third a^3, &c. If a then is greater than 1, the remainder continually increases; which shows, that the farther the division is carried, the greater is the quantity, either positive or negative, which ought to be added to the quotient. The series is, therefore, *diverging* instead of *converging.*

But if a be *less* than 1, the remainders, a, a^2, a^3, &c. will continually decrease. For powers are raised by multiplication; and if the multiplier be less than a unit, the product will be less than the multiplicand. (Art. 85.) If a be taken equal to $\frac{1}{2}$, then by Art. 237,

$$a^2=\tfrac{1}{4},\ a^3=\tfrac{1}{8},\ a^4=\tfrac{1}{16},\ a^5=\tfrac{1}{32},\ \&c.$$

and we have

$$\frac{1}{1-\frac{1}{2}}=\frac{1}{\frac{1}{2}}=2=1+\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{8}+\tfrac{1}{16}+\tfrac{1}{32}+\tfrac{1}{64},\ \&c.$$

Here the *two* first terms $=1+\frac{1}{2}$, which is less than 2, by $\frac{1}{2}$;
the *three* first $=1+\frac{3}{4}$, less than 2, by $\frac{1}{4}$;
the *four* first $=1+\frac{7}{8}$, less than 2, by $\frac{1}{8}$;

So that the farther the series is carried, the nearer it approaches to the value of the given fraction, which is equal to 2.

2. If $\frac{1}{1+a}$ be expanded, the series will be the same as that from $\frac{1}{1-a}$, except that the terms which consist of the *odd* powers of a will be *negative.*

So that $\frac{1}{1+a}=1-a+a^2-a^3+a^4-a^5+a^6$, &c.

3. Reduce $\dfrac{h}{a-b}$ to an infinite series.

$$
\begin{array}{l}
a-b)h \quad \left(\dfrac{h}{a}+\dfrac{bh}{a^2}+\dfrac{b^2h}{a^3},\ \&c.\right. \\
\qquad h-\dfrac{bh}{a} \\
\hline
\qquad * \quad \dfrac{bh}{a},\ \&c.
\end{array}
$$

If the operation be continued in the same manner, we shall obtain the series,

$$\frac{h}{a}+\frac{bh}{a^2}+\frac{b^2h}{a^3}+\frac{b^3h}{a^4}+\frac{b^4h}{a^5},\ \&c.$$

in which the exponents of b and of a increase regularly by 1.

4. Reduce $\dfrac{1+a}{1-a}$ to an infinite series.

Ans. $1+2a+2a^2+2a^3+2a^4$, &c.

513. Another method of forming an infinite series is, *by extracting the root of a compound surd.*

Ex. 1. Reduce $\sqrt{a^2+b^2}$ to an infinite series, by extracting the square root according to the rule in Art. 508.

$$
\begin{array}{ll}
 & a^2+b^2\left(a+\dfrac{b^2}{2a}-\dfrac{b^4}{8a^3}+\dfrac{b^6}{16a^5},\ \&c.\right. \\
 & a^2 \\
\hline
2a+\dfrac{b^2}{2a}\Big) & * \quad b^2 \\
 & b^2+\dfrac{b^4}{4a^2} \\
\hline
2a+\dfrac{b^2}{a}-\dfrac{b^4}{8a^3}\Big) & * -\dfrac{b^4}{4a^2},\ \&c.
\end{array}
$$

2. $\sqrt{a^2-b^2}=a-\dfrac{b^2}{2a}-\dfrac{b^4}{8a^3}-\dfrac{b^6}{16a^5}$, &c.

3. $\sqrt{2}=\sqrt{1+1}=1+\frac{1}{2}-\frac{1}{8}+\frac{1}{16}$, &c.

4. $\sqrt{1+x}=1+\dfrac{x}{2}-\dfrac{x^2}{8}+\dfrac{x^3}{16}-\dfrac{5x^4}{128}$, &c.

514. A binomial which has a negative or fractional exponent, may be expanded into an infinite series by the *binomial theorem.* See Arts. 503, 505, and the examples at the end of Sec. xvi.

Indeterminate Co-efficients.

515. A fourth method of expanding an algebraic expression, is by *assuming* a series, with *indeterminate co-efficients;* and afterwards finding the value of these co-efficients.

If the series, to which any algebraic expression is assumed to be equal, be

$$A+Bx+Cx^2+Dx^3+Ex^4, \text{ \&c.}$$

let the equation be reduced to the form in which one of the members is 0. (Art. 183.) Then if such values be assigned to A, B, C, &c. that the co-efficients of the several powers of x, as well as the aggregate of the terms into which x *does not* enter, shall be *each equal* to 0; it is evident that the *whole* will be equal to 0, and that, upon this condition, the equation is correctly stated.

The values of A, B, C, &c. are determined, by reducing the equations in which they are respectively contained.

Ex. 1. Expand into a series $\dfrac{a}{c+bx}$.

Assume $\dfrac{a}{c+bx}=A+Bx+Cx^2+Dx^3+Ex^4$, &c.

Then multiplying by the denominator $c+bx$, and transposing a, we have

$$0=(Ac-a)+(Ab+Bc)x+(Bb+Cc)x^2+(Cb+Dc)x^3, \text{ \&c.}$$

Here it is evident, that if $(Ac-a)$, $(Ab+Bc)$, $(Bb+Cc)$, &c. be made each equal to 0, the several parts of the second member of the equation will vanish, (Art. 106,) and the *whole* will be equal to 0, as it ought to be, according to the assumption which has been made.

Reducing the following equations,

$$Ac-a=0, \qquad \text{we have} \quad A=\frac{a}{c},$$

$$Ab+Bc=0, \qquad B=-\frac{b}{c}A,$$

$$Bb+Cc=0, \qquad C=-\frac{b}{c}B,$$

$$Cb+De=0, \qquad D=-\frac{b}{c}C,$$
&c. &c.

That is, each of the co-efficients, C, D, and E, is equal to the preceding one multiplied into $-\frac{b}{c}$. We have therefore,

$$\frac{a}{c+bx}=\frac{a}{c}-\frac{ab}{c^2}x+\frac{ab^2}{c^3}x^2-\frac{ab^3}{c^4}x^3+\frac{ab^4}{c^5}x^4, \text{ \&c.}$$

2. Expand into a series $\dfrac{a+bx}{d+hx+cx^2}$.

Assume $\dfrac{a+bx}{d+hx+cx^2}=A+Bx+Cx^2+Dx^3$, &c.

Then multiplying by the denominator of the fraction, and transposing $a+bx$, we have $0=(Ad-a)+(Bd+Ah-b)x+(Cd+Bh+Ac)x^2+(Dd+Ch+Bc)x^3$, &c.

Therefore, $A=\dfrac{a}{d}$, $\qquad C=-\dfrac{h}{d}B-\dfrac{c}{d}A$,

$B=-\dfrac{h}{d}A+\dfrac{b}{d}$, $\qquad D=-\dfrac{h}{d}C-\dfrac{c}{d}B$.

And $\dfrac{a+bx}{d+hx+cx^2}=\dfrac{a}{d}-\left(\dfrac{h}{d}A-\dfrac{b}{d}\right)x-\left(\dfrac{h}{d}B+\dfrac{c}{d}A\right)x^2$, &c.

3. Expand into a series $\dfrac{1+2x}{1-x-x^2}$.

Ans. $1+3x+4x^2+7x^3+11x^4+18x^5+29x^6$, &c.

In which, the co-efficient of each of the powers of x, is equal to the *sum* of the co-efficients of the *two* preceding terms.

4. Expand into a series $\dfrac{d}{b-ax}$.

Ans. $\dfrac{d}{b}\left(1+\dfrac{ax}{b}+\dfrac{a^2x^2}{b^2}+\dfrac{a^3x^3}{b^3}+\dfrac{a^4x^4}{b^4}, \text{ \&c.}\right)$

5. Expand into a series $\dfrac{1-x}{1-2x-3x^2}$.

Ans. $1+x+5x^2+13x^3+41x^4+121x^5+365x^6$, &c.

6. Expand into a series $\dfrac{1}{1-x-x^2+x^3}$.

Ans. $1+x+2x^2+2x^3+3x^4+3x^5+4x^6+4x^7$, &c.

7. Expand $\dfrac{a}{1-bx}$. 8. Expand $\dfrac{1-x}{1-5x+6x^2}$.

9. Expand $\dfrac{a+bx}{(1-dx)^2}$. 10. Expand $\dfrac{1+x}{(1-x)^2}$.

In the preceding examples, the series assumed contains x and its powers, with positive exponents, rising regularly from the second term. But this form is not *universally* applicable. There are some algebraic expressions to which it can not be applied, without leading to absurd results; shewing that the method is not fitted to expand correctly these particular results.

Summation of Series.

516. Though an infinite series consists of an unlimited number of terms, yet, in many cases, it is not difficult to find what is called the *sum of the terms;* that is, a quantity which differs less, than by any assignable quantity, from the value of the whole. This is also called the *limit* of the series. Thus the decimal 0.33333, &c. may come infinitely near to the vulgar fraction $\frac{1}{3}$, but never can exceed it, nor, indeed, exactly equal it. See Arts. 463, 4. Therefore $\frac{1}{3}$ is the limit of 0.33333, &c. that is, of the series

$$\tfrac{3}{10}+\tfrac{3}{100}+\tfrac{3}{1000}+\tfrac{3}{10000}+\tfrac{3}{100000}, \text{ \&c.}$$

If the number of terms be supposed infinitely great, the difference between their sum and $\frac{1}{3}$, will be infinitely small.

517. The sum of an infinite series whose terms decrease by a common divisor, may be found, by the rule for the sum of a series in *geometrical progression.* (Art. 452.) According to this, $S=\dfrac{rz-a}{r-1}$, that is, the sum of the series is found by multiplying the greatest term into the ratio, subtracting the least term, and dividing by the ratio less 1. But, in an infinite series decreasing, the least term is infinitely small. It may be neglected therefore as of no comparative value. (Art. 466.) The formula will then become,

$$S=\frac{rz-0}{r-1} \text{ or } S=\frac{rz}{r-1}.$$

Ex. 1. What is the sum of the infinite series

$$\tfrac{3}{10}+\tfrac{3}{100}+\tfrac{3}{1000}+\tfrac{3}{10000}+\tfrac{3}{100000}, \text{ \&c.}$$

Here the first term is $\frac{3}{10}$, and the ratio is 10.

$$\text{Then } S=\frac{rz}{r-1}=\frac{10\times\frac{3}{10}}{10-1}=\frac{3}{9}=\frac{1}{3}, \text{ the answer.}$$

2. What is the sum of the infinite series

$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}$, &c.? Ans. $S=\frac{rz}{r-1}=\frac{2\times 1}{2-1}=2$.

3. What is the sum of the infinite series

$1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}$, &c.? Ans. $\frac{3}{2}=1+\frac{1}{2}$.

Recurring Series.

518. When a series is so constituted, that a certain number of contiguous terms, taken in any part of the series, have a given relation to the term immediately succeeding, it is called a *recurring series;* as any one of the following terms may be found, by *recurring* to those which precede.

Thus in the series $1+3x+4x^2+7x^3+11x^4+18x^5$, &c., the *sum* of the co-efficients of any two contiguous terms, is equal to the co-efficient of the following term.

In the series $1+2x+3x^2+4x^3+5x^4+6x^5$, &c., each term, after the second, is equal to $2x$ multiplied by the term immediately preceding, $-x^2$ multiplied by the term next preceding. The co-efficients of x and x^2, that is, $+2-1$, constitute what is called the *scale of relation.*

In the series $1+4x+6x^2+11x^3+28x^4+63x^5$, &c., any *three* contiguous terms have a constant relation to the succeeding term. The *scale of relation* is $2-1+3$; so that each term, after the third, is equal to $2x$ into the term immediately preceding, $-x^2$ into the term next preceding, $+3x^3$ into the third preceding term.

Let any recurring series be expressed by

$$A+B+C+D+E+F, \text{ \&c.}$$

If the law of progression depends upon *two* contiguous terms and the scale of relation consists of two parts, m and n,

Then $C=Bmx+Anx^2$, the third term,
$D=Cmx+Bnx^2$, the fourth.
$E=Dmx+Cnx^2$, the fifth,
&c. &c.

If the law of progression depends on *three* contiguous terms, and the scale of relation is $m+n+r$,

Then $D=Cmx+Bnx^2+Arx^3$, the fourth term,
$E=Dmx+Cnx^2+Brx^3$, the fifth,
$F=Emx+Dnx^2+Crx^3$, the sixth,
&c. &c.

If the law of progression depends on *more than three terms*, the succeeding terms are derived from them in a similar manner.

519. In any recurring series, the *scale of relation*, if it consists of *two* parts, may be *found*, by reducing the equations expressing the values of two of the terms; if it consists of *three* parts, it may be found by reducing the equations expressing the values of three terms, &c. As the scale of relation is the same, whatever be the value of x in the series, the reduction may be rendered more simple, by making $x=1$.

Taking then the fourth and fifth terms, in the first example above, and making $x=1$, we have

$$\left.\begin{array}{l} D=Cm+Bn \\ E=Dm+Cn \end{array}\right\} \text{ to find the values of } m \text{ and } n.$$

These reduced, give

$$m=\frac{DC-BE}{CC-BD} \qquad n=\frac{CE-DD}{CC-BD}.$$

In the series $\left\{\begin{array}{l} A \quad B \quad C \quad D \quad E \quad F \\ 1+3x+5x^2+7x^3+9x^4+11x^5, \text{ \&c.} \end{array}\right.$

Making $x=1$, we have

$$m=\frac{7\times5-3\times9}{5^2-3\times7}=2. \qquad n=\frac{5\times9-7^2}{5^2-3\times7}=-1.$$

Therefore, the scale of relation is $2-1$.

To know whether the law of progression depends on *two*, *three*, or *more* terms; we may first make a trial of two terms; and if the scale of relation thus found, does not correspond with the given series, we may try three or more terms. Or if we begin with a number of terms greater than is necessary, one or more of the values found will be 0, and the others will constitute the true scale of relation.

520. When the scale of relation of a decreasing recurring series is known, the *sum of the terms* may be found.

Let $\begin{cases} A \quad B \quad C \quad D \quad E \quad F \\ a+bx+cx^2+dx^3+ex^4+fx^5, \text{ \&c.} \end{cases}$

be a recurring series, of which the scale of relation is $m+n$.

Then $A=$ the first term, $B=$ the second,
$C=B\times mx+A\times nx^2$, the third,
$D=C\times mx+B\times nx^2$, the fourth,
$E=D\times mx+C\times nx^2$, the fifth.
&c. &c.

Here mx is multiplied into every term, except the first and the last; and nx^2 into every term except the two last. If the series be infinitely extended, the last terms may be neglected, as of no comparative value, (Art. 466,) and if $S=$ the sum of the terms, we have

$S=A+B+mx\times(B+C+D, \text{ \&c.})+nx^2\times(A+B+C, \text{ \&c.})$

But $S-A=B+C+D$, &c. And $S=A+B+C$, &c.

Therefore $S=A+B+mx\times(S-A)+nx^2\times S$.

Reducing this equation, we have

$$S=\frac{A+B-Amx}{1-mx-nx^2}.$$

Ex. 1. What is the sum of the infinite series

$$1+6x+12x^2+48x^3+120x^4, \text{ \&c.?}$$

The scale of relation will be found to be $1+6$.

Then $A=1$, $B=6x$, $m=1$, $n=6$.

The series therefore $=\dfrac{1+5x}{1-x-6x^2}$.

2. What is the sum of the infinite series

$1+3x+4x^2+7x^3+11x^4+18x^5+29x^6$, &c.? Ans. $\dfrac{1+2x}{1-x-x^2}$

3. What is the sum of the infinite series

$$1+x+5x^2+13x^3+41x^4+121x^5+365x^6, \text{ \&c.?}$$

Ans. $\dfrac{1-x}{1-2x-3x}$.

4. What is the sum of the infinite series

$1+2x+3x^2+4x^3+5x^4$, &c.? Ans. $\dfrac{1+2x-2x}{1-2x+x^2}=\dfrac{1}{(1-x)^2}$

Method of Differences.

521. In the Summation of Series, the object of inquiry is not, always, to determine the value of the *whole* when infinitely extended; but frequently, to find the sum of *a certain number of terms*. If the series is an *increasing* one, the sum of all the terms is infinite. But the value of a limited number of terms may be accurately determined. And it is frequently the case, that a part of a *decreasing* series, may be more easily summed than the whole. A moderate number of terms at the commencement of the series, if it converges rapidly, may be a near approximation to the amount of the whole, when indefinitely extended.

One of the methods of determining the value of a limited number of terms, depends on finding the several *orders of differences* belonging to the series. The differences between the terms themselves, are called the *first order* of differences; the differences of these differences, the *second order*, &c. In the series,

1, 8, 27, 64, 125, &c.

by subtracting each term from the next, we obtain the first order of differences,

7, 19, 37, 61, &c.

and taking each of these from the next, we have the second order,

12, 18, 24, &c.

Proceeding in this manner with the series

$a, b, c, d, e, f,$ &c.

we obtain the following ranks of differences,

1st. Diff. $b-a,\ c-b,\ d-c,\ e-d,\ f-e,$ &c.

2d. Diff. $c-2b+a,\ d-2c+b,\ e-2d+c,\ f-2e+d,$ &c.

3d. Diff. $d-3c+3b-a,\ e-3d+3c-b,\ f-3e+3d-c,$ &c.

4th. Diff. $e-4d+6c-4b+a,\ f-4e+6d-4c+b,$ &c.

5th. Diff. $f-5e+10d-10c+5b-a,$ &c.

&c &c.

In these expressions, each difference, here pointed off by commas, though a compound quantity, is called a *term*. Thus the first term in the first rank is $b-a$; in the second, $c-2b+a$; in the third, $d-3c+3b-a$; &c. The first *terms*, in the several orders, are those which are principally employed in

investigating and applying the method of differences. It will be seen, that in the preceding scheme of the successive differences, the *co-efficients* of the first term,

In the second rank, are	1, 2, 1;
In the third,	1, 3, 3, 1;
In the fourth,	1, 4, 6, 4, 1;
In the fifth,	1, 5, 10, 10, 5, 1;

Which are the same, as the co-efficients in the *powers of binomials.* (Art. 488.) Therefore, the co-efficients of the first term in the nth order of differences, (Art. 493,) are

$$1,\ n,\ n\times\frac{n-1}{2},\ n\times\frac{n-1}{2}\times\frac{n-2}{3},\ \text{\&c.}$$

522. For the purpose of obtaining a general expression for *any term* of the series a, b, c, d, &c. let D', D'', D''', D'''', &c. represent the *first terms,* in the first, second, third, fourth, &c. orders of differences.

$$\begin{aligned} \text{Then}\quad D' &= b-a,\\ D'' &= c-2b+a,\\ D''' &= d-3c+3b-a,\\ D'''' &= e-4d+6c-4b+a,\\ \text{\&c.}\quad & \quad\text{\&c.} \end{aligned}$$

Transposing and reducing these, we obtain the following expressions for the terms of the original series, a, b, c, d, &c.

The second term	$b=a+D'$,
The third,	$c=a+2D'+D''$,
The fourth,	$d=a+3D'+3D''+D'''$,
The fifth,	$e=a+4D'+6D''+4D'''+D''''$.

Here the co-efficients observe the same law, as in the *powers of a binomial;* with this difference, that the co-efficients of the nth term of the series, are the co-efficients of the $(n-1)$th power of a binomial.

Thus the co-efficients of the fifth term are 1, 4, 6, 4, 1; which are the same as the co-efficients of the *fourth* power of a binomial. Substituting, then, $n-1$ for n, in the formula for the co-efficients of an involved binomial, (Art. 493,) and applying the co-efficients thus obtained to D', D'', D''', D'''', &c. as in the preceding equations, we have the following general expression, for the nth term of the series, a, b, c, d, &c.

The nth term

$$=a+(n-1)D'+(n-1)\frac{n-2}{2}D''+(n-1)\frac{n-2}{2}\times\frac{n-3}{3}D''', \text{ \&c.}$$

When the differences, after a few of the first orders, become 0, any term of the series is easily found.

Ex. 1. What is the nth term of the series 1, 3, 6, 10, 15, 21?

Proposed series,	1, 3, 6, 10, 15, 21, &c.
First order of differences,	2, 3, 4, 5, 6, &c.
Second do.	1, 1, 1, 1, &c.
Third do.	0, 0, 0.

Here $a=1$, $D'=2$, $D''=1$, $D'''=0$.

Therefore the nth term $=1+(n-1)2+(n-1)\frac{n-2}{2}$.

The 20th term $=1+38+171=210$. The 50th $=1275$.

2. What is the 20th term of the series $1^3, 2^3, 3^3, 4^3, 5^3$, &c.?

Proposed series,	1, 8, 27, 64, 125, &c.
First order of differences,	7, 19, 37, 61, &c.
Second do.	12, 18, 24, &c.
Third do.	6, 6, &c.

Here $D'=7$, $D''=12$, $D'''=6$.

Therefore the 20th term $=8000$.

3. What is the 12th term of the series 2, 6, 12, 20, 30, &c.? Ans. 156.

4. What is the 15th term of the series $1^2, 2^2, 3^2, 4^2, 5^2, 6^2$, &c.? Ans. 225.

523. To obtain an expression for the *sum* of any number of terms of a series a, b, c, d, &c. let one, two, three, &c. terms be successively added together, so as to form a *new series*,

$$0,\quad a,\quad a+b,\quad a+b+c,\quad a+b+c+d, \text{ \&c.}$$

Taking the differences in this, we have

1st Diff. $a,\quad b,\quad c,\quad d,\quad e,\quad f$, &c.

2d Diff. $b-a,\ c-b,\ d-c,\ e-d,\ f-e$, &c.

3d Diff. $c-2b+a,\ d-2c+b,\ e-2d+c,\ f-2e+d$, &c.

4th Diff. $d-3c+3b-a,\ e-3d+3c-b,\ f-3e+3d-c$, &c.

&c. &c.

Here it will be observed that the *second* rank of differences in the *new* series, is the same as the *first* rank in the *original* series a, b, c, d, e, &c. and generally, that the $(n+1)$th rank in the new series is the same as the nth rank in the original series. If, as before, $D'=$ the first term of the first differences in the original series, and $d'=$ the first term of the first differences in the new series;

Then $\quad d'=a, \quad d''=D', \quad d'''=D'', \quad d''''=D'''$, &c.

Taking now the formula (Art. 522,)

$$a+(n-1)D'+(n-1)\frac{n-2}{2}D''+(n-1)\frac{n-2}{2}\times\frac{n-3}{3}D'''+\text{&c.}$$

which is a general expression for the nth term of a series in which the first term is a; applying it to the new series, in which the first term is 0, and substituting $n+1$ for n, we have

$$0+nd'+n\frac{n-1}{2}d''+n\frac{n-1}{2}\times\frac{n-2}{3}d'''+n\frac{n-1}{2}\times\frac{n-2}{3}\times\frac{n-3}{4}d''''+\text{&c.}$$

Or $$na+n\frac{n-1}{2}D'+n\frac{n-1}{2}\times\frac{n-2}{3}D''+n\frac{n-1}{2}\times\frac{n-2}{3}\times\frac{n-3}{4}D'''+\text{&c.}$$

Which is a general expression for the $(n+1)$th term of the series

$$0, \quad a, \quad a+b, \quad a+b+c, \quad a+b+c+d, \text{ &c.}$$

or the nth term of the series

$$a, \quad a+b, \quad a+b+c, \quad a+b+c+d, \text{ &c.}$$

But the nth term of the latter series, is evidently the *sum* of n terms of the series, a, b, c, d, &c. Therefore *the general expression for the sum of n terms of a series of which a is the first term, is*

$$na+n\frac{n-1}{2}D'+n\frac{n-1}{2}\times\frac{n-2}{3}D''+n\frac{n-1}{2}\times\frac{n-2}{3}\times\frac{n-3}{4}D'''+\text{&c.}$$

Ex. 1. What is the sum of n terms of the series of *odd numbers*, 1, 3, 5, 7, 9, &c.?

Series proposed,	1, 3, 5, 7, 9, &c.
First order of difference,	2, 2, 2, 2, &c.
Second order of difference,	0, 0, 0.

Here $a=1, \quad D'=2, \quad D''=0.$

Therefore the sum of n terms $=n+n\frac{n-1}{2}\times2=n^2.$

That is, the *sum* of the terms is equal to the *square of the number* of terms. See Art. 441.

2. What is the sum of n terms of the series

$$1^2,\ 2^2,\ 3^2,\ 4^2,\ 5^2,\ \&c.?$$

Here $a=1$, $D'=3$, $D''=2$, $D'''=0$.

Therefore n terms $=\frac{1}{6}(2n^3+3n^2+n)=\frac{1}{6}n(n+1)\times(2n+1)$.

Thus the sum of 20 terms $=2870$.

3. What is the sum of n terms of the series

$$1^3,\ 2^3,\ 3^3,\ 4^3,\ \&c.?$$

Here $a=1$, $D'=7$, $D''=12$, $D'''=6$, $D''''=0$.

Therefore n terms $=\frac{1}{4}(n^4+2n^3+n^2)=(\frac{1}{2}n\times\overline{n+1})^2$.

Thus the sum of 50 terms $=1625625$.

4. What is the sum of n terms of the series

2, 6, 12, 20, 30, &c.? Ans. $\frac{1}{3}n(n+1)\times(n+2)$.

5. What is the sum of 20 terms of the series

1, 3, 6, 10, 15, &c.?

6. What is the sum of 12 terms of the series

$$1^4,\ 2^4,\ 3^4,\ 4^4,\ 5^4,\ \&c.?$$

Continued Fractions.

524. One form of series is given in the algebraic expressions which are denominated *continued fractions*. Among the uses to which these are applied, one of the principal is in approximating to the values of fractions which are expressed in large numbers. If for the purpose of reducing the fraction $\frac{1393}{972}$ to lower terms, we divide it by the denominator, we obtain the quotient 1, *and a remainder*, which placed over the divisor, gives the fraction $\frac{421}{972}$

Dividing by the numerator, we have $\frac{1}{2}+$ *the fraction* $\frac{130}{421}$

Dividing again, we have $\frac{1}{3}+$ *the fraction* $\frac{31}{130}$

&c. &c.

Here, each of the fractions, after the first, is divided by its numerator, which is *the remainder* in the preceding division; the preceding divisor now becoming a dividend, as in the process for finding the greatest common measure of two quantities, (Art. 476.) Each of the denominators, after the

division, consists of *two parts*, an integer and a fraction less than a unit. The results may be arranged as follows.

$$\frac{1393}{972}=1+\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{4+\cfrac{1}{5+\cfrac{1}{6}}}}}.\quad \text{Generally, } \frac{a}{b}=p+\cfrac{1}{q+\cfrac{1}{r+\cfrac{1}{s}}}\ \&c.\quad \text{Or } \frac{a}{b}=\cfrac{1}{p+\cfrac{1}{q+\cfrac{1}{r+\cfrac{1}{s}}}}\ \&c.$$

If the numerator of the original fraction is greater than the denominator, the series will begin with an integer, as in the first two forms above. But if the numerator is less than the denominator, the series will begin with a fraction, as in the third form.

525. *A continued fraction*, then, *is one whose denominator is a whole number and a fraction; the denominator of the latter being also a whole number and a fraction;* the formula being continued, in a series of similar fractions.

To throw a common fraction into the form of a continued fraction, we have, then, the following rule; *Divide the greater term of the given fraction by the less, and this divisor and the following ones, by the several remainders successively;* as in finding the greatest common measure of two numbers.

The fractions $\frac{1}{p}$, $\frac{1}{q}$, $\frac{1}{r}$, &c. taken by themselves are called *integral* fractions, as their denominators are whole numbers, (Art. 524.) Each integral fraction is added, not to the whole preceding one, but to its *denominator* only.

526. The separate expressions $\frac{1}{p}$, $\cfrac{1}{p+\cfrac{1}{q}}$, $\cfrac{1}{p+\cfrac{1}{q+\cfrac{1}{r}}}$, &c.

are called *approximating* or *converging* fractions, as each, in succession, gives a nearer approximation than the preceding one, to the exact value of the original fraction $\frac{a}{b}$. They are not independent expressions, as each following one is only an extension of the preceding one. In the formula already given, $\cfrac{1}{p+\cfrac{1}{q}}$ is nearer the value of $\frac{a}{b}$ than $\frac{1}{p}$. For an *addition* must be made to the denominator of $\frac{1}{p}$, to render it

equivalent to $\frac{a}{b}$. (Art. 524.) And a *part* of this addition is made in $\cfrac{1}{p+\cfrac{1}{q}}$. Again, the *third* converging fraction $\cfrac{1}{p+\cfrac{1}{q+\cfrac{1}{r}}}$ is nearer in value to $\frac{a}{b}$, than the *second* $\cfrac{1}{p+\cfrac{1}{q}}$. For an addition must be made to the denominator of this, to render it equivalent to $\frac{a}{b}$; and a part of this addition is made in $\cfrac{1}{p+\cfrac{1}{q+\cfrac{1}{r}}}$. The same proof may be extended to each successive converging fraction. That portion of the original quantity which remains unexhausted by the continued fraction is diminished, by each successive addition of an integral fraction.

527. The *first* converging fraction $\frac{1}{p}$ *exceeds* the original $\frac{a}{b}$. For the denominator p is *too small*, and this renders the value of the fraction too great. (Art. 143, cor.) But the *second* converging fraction $\cfrac{1}{p+\cfrac{1}{q}}$ is *less* than $\frac{a}{b}$; for the denominator q is too small, and this renders $\frac{1}{q}$, and of course $p+\frac{1}{q}$ too great; and therefore the value of $\cfrac{1}{p+\cfrac{1}{q}}$ too small. In the same manner, it may be shown, that the *third* converging fraction is too great, the *fourth* too small, &c. that is, Of several successive converging fractions, *those which are counted by* ODD *numbers are* TOO LARGE, *while those which are counted by* EVEN *numbers are* TOO SMALL. Each succeeding one, however, comes nearer than the preceding one, to the value of the original fraction. It follows, that the exact value of this is *between* two consecutive fractions, one of which is too great, and the other too small.

528. When the numerator and denominator of the given fraction are *commensurable*, the series of continued fractions

derived from it *will terminate;* as the process of forming it, after several divisions, will give a quotient *without a remainder*. But if the terms of the original fraction are *incommensurable,* the divisions may be continued without limit, forming an infinite series.

Ex. 1. What is the continued fraction corresponding to $\frac{235}{1683}$. Ans. The successive integral fractions are $\frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{1}{2}, \frac{1}{3}$. The last of these leaves no remainder. The series, therefore, terminates here; and the last continued fraction is exactly equal to the vulgar fraction from which it is derived.

2. What is the continued fraction formed from $\frac{1103}{887}$?

Ans. The integral fractions are $1+\frac{1}{4}, \frac{1}{9}, \frac{1}{2}, \frac{1}{1}, \frac{1}{1}, \frac{1}{4}$.

3. What are the integral fractions derived from $\frac{1461}{59}$?

Ans. $24+\frac{1}{1}, \frac{1}{3}, \frac{1}{4}, \frac{1}{1}, \frac{1}{2}$.

529. The values of other quantities besides fractions, particularly of surd quantities, may be expressed in the form of continued fractions. If x be a quantity which can not be exactly stated in whole numbers, let the greatest integer contained in it be a. Then $x=a+$ a fraction *less* than a unit. Let the latter be $\frac{1}{y}$. Its denominator y must be *greater* than 1. (Art. 141.) Let the greatest integer contained in it be b, and the remaining fraction be $\frac{1}{y'}$. Then $y=b+\frac{1}{y'}$. In like manner, let the greatest integer contained in y' be b', and the remaining fraction be $\frac{1}{y''}$, &c. We have then $x=a+\frac{1}{y}$, $y=b+\frac{1}{y'}$, $y'=b'+\frac{1}{y''}$, $y''=b''+\frac{1}{y'''}$, &c. Then substituting for y, y', y'', &c. their several values, we have

$$x=a+\frac{1}{y}=a+\cfrac{1}{b+\cfrac{1}{y'}}=a+\cfrac{1}{b+\cfrac{1}{b'+\cfrac{1}{y''}}}=a+\cfrac{1}{b+\cfrac{1}{b'+\cfrac{1}{b''+\cfrac{1}{y'''}}}}, \text{ \&c.}$$

Ex. The square root of 2 is 1.414213+ in decimals, or $\frac{1414213+}{1000000}$. If this be divided, according to the rule in Art. 525, the first six integral fractions will be each $\frac{1}{2}$, forming the continued fraction $\sqrt{2}=1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2}}}$, &c.

530. A continued fraction may be reduced to a vulgar fraction, and its *value* found, by the common rules for the reduction of fractions; and by comparing several converging fractions, a rule is obtained, by which, if any contiguous ones in the series are known, the value of the next may he found. Of the three following converging fractions,

The 1st,

$$\frac{1}{p} \qquad =\frac{1}{p} \qquad \text{Or } a+\frac{1}{p} \qquad =\frac{ap+1}{p}$$

The 2d,

$$\cfrac{1}{p+\cfrac{1}{q}} =\frac{q}{pq+1} \text{ (Art. 171.) } a+\cfrac{1}{p+\cfrac{1}{q}} =\frac{apq+a+q}{pq+1}$$

The 3d,

$$\cfrac{1}{p+\cfrac{1}{q+\cfrac{1}{r}}}=\frac{qr+1}{(pq+1)r+p} \qquad a+\cfrac{1}{p+\cfrac{1}{q+\cfrac{1}{r}}}=\frac{(apq+a+q)r+ap+1}{(pq+1)r+p}$$

Here it will be seen, that the numerator of the *third* converging fraction is equal to the product of the numerator of the preceding fraction, into the denominator of the third *integral* fraction, added to the numerator of the first converging fraction; and

The denominator is equal to the product of the denominator of the preceding fraction into the denominator of the third integral fraction, added to the denominator of the first converging fraction.

To show that this is a general law, let the three fractions above be represented by

$$\frac{N}{D}, \frac{N'}{D'}, \text{ and } \frac{N''}{D''}.$$

Then, as has just been shown, $\frac{N''}{D''}=\frac{N'r+N}{D'r+D}$.

Let the *fourth* converging fraction $\frac{N'''}{D'''}$ be now added, its last integral fraction being $\frac{1}{s}$. The only change here made, is in adding $\frac{1}{s}$ to r. To obtain the value, then, we have only to substitute $r+\frac{1}{s}$ for r, making

$$\frac{N'''}{D'''}=\frac{N'(r+\frac{1}{s})+N}{D'(r+\frac{1}{s})+D}=\frac{(N'r+N)s+N'}{(D'r+D)s+D'}$$

But $\frac{N'r+N=N''}{D'r+D=D''}$. Therefore $\frac{N'''}{D'''}=\frac{N''s+N'}{D''s+D'}$.

And as one converging fraction after another is formed, by merely adding a new integral fraction, we have this general law;

Of any three consecutive converging fractions, *the numerator of the last is equal to the product of the numerator of the preceding one, into the denominator of the last integral fraction, added to the numerator of the first of the three;* and

The denominator of the last is equal to the product of the denominator of the preceding one, into the denominator of the last integral fraction, added to the denominator of the first of the three.

531. The difference of the numerators of two contiguous converging fractions is either $+1$ or -1; and the denominator of this difference is the product of the denominators of the two fractions.

Thus $\frac{1}{p}-\frac{q}{pq+1}=\frac{pq+1-pq}{p(pq+1)}=\frac{1}{p(pq+1)}$,

And $\frac{q}{pq+1}-\frac{qr+1}{(pq+1)r+p}=\frac{-1}{(pq+1)\times\overline{(pq+1)r+p}}$.

And if *any* three consecutive converging fractions be taken, the difference of the numerators of the first two, when reduced to a common denominator, is the same, with a contrary sign, as the difference of the numerators of the other two.

Let $\frac{N}{D}, \frac{N'}{D'}, \frac{N''}{D''}$, be three consecutive converging fractions.

Then $\frac{N}{D} - \frac{N'}{D'} = \frac{ND' - N'D}{DD'}$

And $\frac{N'}{D'} - \frac{N''}{D''} = \frac{N'D'' - N''D'}{D'D''}$

But $N'' = N'r + N$. (Art. 530.) And $D'' = D'r + D$.

By substitution, then, $\frac{N'}{D'} - \frac{N''}{D''} = \frac{N'(D'r+D) - D'(N'r+N)}{D'D''}$

This, when reduced, gives $\frac{N'}{D'} - \frac{N''}{D''} = \frac{-ND' + N'D}{D'D''}$.

Shewing the difference between the last two numerators to be the same, with the contrary sign, as the difference between the first two.

As in the example at the beginning of this article, the difference of the first two enumerators is $+1$, and the difference of the last two is -1; it follows from this, and the general law now proved, that the difference must be alternately $+1$ and -1, throughout the series.

532. *The excess or deficiency of any converging fraction is less than a unit divided by the square of its denominator,* when reduced to a vulgar fraction.

Let $\frac{N}{D}, \frac{N'}{D'}, \frac{N''}{D''}$, be three contiguous fractions, and the last integral fraction be $\frac{1}{r}$, r being a whole number. (Art. 525.)

The difference between the values of $\frac{N'}{D'}$ and $\frac{N''}{D''}$, is $\frac{\pm 1}{D'D''}$. (Art. 531.) As the value of the original quantity from which these are derived is *between* the two, (Art. 527,) it follows that the excess or deficiency of either is *less* than $\frac{\pm 1}{D'D''}$. But as $D'' = D'r + D$, (Art. 530,) D'' is *greater* than D', and of course, $D'D''$ is greater than $D'D'$. Therefore $\frac{\pm 1}{D'D''}$ is less than $\frac{\pm 1}{D'^2}$. (Art. 143, cor.) And as the excess or defi-

ciency of $\frac{N'}{D'}$ is less than $\frac{\pm 1}{D'D''}$, still more is it less than $\frac{\pm 1}{D'^2}$.

As the denominators of the continued fractions go on increasing, while the series is extended, we have proof here, as well as in Art. 526, that the small error which each contains is *continually diminishing;* and that the formulas are approaching nearer and nearer to the exact value of the quantity from which they are derived.

533. One of the applications of the principles of continued fractions is in finding eligible numbers to express nearly the ratio of *the diameter of a circle to its circumference.* This, in decimals, is $3.14159+$, or $\frac{314159}{100000}+$. If we divide according to the rule in Art. 525, we obtain the following integral fractions, $\frac{3}{1}$, $\frac{1}{7}$, $\frac{1}{15}$, $\frac{1}{1}$, &c. Reducing the converging fractions formed with these, (Art. 530,) we have the following vulgar fractions, $\frac{3}{1}$, $\frac{22}{7}$, $\frac{333}{106}$, $\frac{355}{113}$, &c. The second of these gives the proximate ratio of the circumference to the diameter 22 to 7, as demonstrated by *Archimedes.* The fourth gives it much more accurately, 355 to 113, the ratio assigned by *Metius.*

Interpolation.

534. Logarithms, trigonometrical sines, tangents, &c. are given in the tables, for the natural series of numbers, or for degrees, minutes, &c. We sometimes have occasion to find the *intermediate* logarithms, sines, &c. for fractional parts of a degree, unit, &c. This is called interpolation. The formula

$$a+(n-1)D'+(n-1)\times\frac{n-2}{2}D''+(n-1)\times\frac{n-2}{2}\times\frac{n-3}{3}D'''+\&c.$$

in Art. 522, answers this purpose.

When the terms of a series a b c d, &c. are numbered 1 2 3, &c. the interval between each of the numbers and the next being considered a unit; it is evident that $n-1$ expresses in portions of this interval, the distance of the nth

number from the first. If then m be substituted for $n-1$, the formula for the nth term becomes

$$a+mD'+m\cdot\frac{m-1}{2}D''+m\cdot\frac{m-1}{2}\cdot\frac{m-2}{3}D'''+\text{\&c.}$$

Ex. 1. Given the logarithms of 140, 141, 142, 143, to find the logarithm of $141\frac{1}{2}$.

The interval here between the numbers is 1. The distance of the number whose logarithm is required from the first is $1\frac{1}{2}=\frac{3}{2}$. The logarithms given in the tables are

For	140,	146128	$D'=3091$	$D''=-22$	
	141,	149219	3069		$D'''=1$
	142,	152288	3048	-21	
	143,	155336			

As the differences of the first order decrease, while the logarithms from which they are derived increase, the differences of the second order are *negative*. (Art. 52.)

The first term a	$=146128$
The second, $mD'=\frac{3}{2}\times3091$	$=4636$
	150764
The third, $m\cdot\frac{m-1}{2}D''=\frac{3}{2}\times\frac{1}{4}\times-22$	$=-8$
The logarithm required is	150756

In this example, the first three terms of the formula give the result required, with a sufficient degree of exactness. And generally, a few terms will answer the purpose. Almost all the logarithms in the tables are *approximations* only; being obtained by series which converge, but do not terminate.

2. Given the logarithmic sines of 20°, 22°, 24°, 26°; viz. 534052, 573575, 609313, 641842, to find the sine of $23\frac{1}{2}$ degrees.

Here the interval between the numbers is 2. The distance m of the required sine from the first number is $1\frac{3}{4}$ of this interval. Therefore $mD'=\frac{7}{4}D'$, $m\cdot\frac{m-1}{2}D''=\frac{7}{4}\times\frac{3}{8}D''$, $m\cdot\frac{m-1}{2}\cdot\frac{m-2}{3}D'''=\frac{7}{4}\times\frac{3}{8}\times-\frac{1}{12}D'''$.

The sine required is 600700.

3. Given the logarithmic tangents of 17°, 18°, 19°, 20°; viz. 485339, 511776, 536972, 561066, to find the tangent of $18°\frac{1}{2}$. Ans. 524520.

4. Given the natural sines of 24°, 25°, 26°, 27°; viz. 40674, 42262, 43837, 45399, to find the natural sine of $25°\frac{2}{3}$. Ans. 43313.

SECTION XIX.

GENERAL PROPERTIES OF EQUATIONS.

ART. **535.** EQUATIONS of any degree may be produced from *simple* equations, by multiplication. The manner in which they are compounded will be best understood, by taking them in that state in which they are all brought on one side by transposition. (Art. 183.) It will also be necessary to assign, to the same letter, different values, in the different simple equations.

Suppose, that in one equation,	$x=2$
And, that in another,	$x=3$
By transposition,	$x-2=0$
And	$x-3=0$
Multiplying them together,	$x^2-5x+6=0$
Next, suppose,	$x-4=0$
And multiplying,	$x^3-9x^2+26x-24=0$
Again suppose,	$x-5=0$
And mult. as before,	$x^4-14x^3+71x^2-154x+120=0$, &c.

Collecting together the products, we have

$$(x-2)(x-3) = x^2-5x+6=0$$
$$(x-2)(x-3)(x-4) = x^3-9x^2+26x-24=0$$
$$(x-2)(x-3)(x-4)(x-5) = x^4-14x^3+71x^2-154x+120=0, \text{ \&c.}$$

That is, the product

of *two* simple equations is a *quadratic* equation;

of *three* simple equations, is a *cubic* equation;

of *four* simple equations, is a *biquadratic*, or an equation of the fourth degree, &c. (Art. 317.)

Or a cubic equation may be considered as the product of a quadratic and a simple equation; a biquadratic, as the product of two quadratic; or of a cubic and a simple equation, &c.

In each case, the *exponent* of the unknown quantity, in the first term, is equal to the degree of the equation; and, in the succeeding terms, it decreases regularly by 1, like the exponent of the leading quantity in the power of a binomial. (Art. 485.)

In a quadratic equation, the exponents are	2, 1,
In a cubic equation,	3, 2, 1,
In a biquadratic,	4, 3, 2, 1, &c.

The *number* of terms, is greater by 1, than the degree of the equation, or the number of simple equations from which it is produced. For besides the terms which contain the different powers of the unknown quantity, there is one which consists of *known* quantities only. The equation is here supposed to be *complete*. But if there are, in the partial products, terms which balance each other, these may *disappear* in the result. (Art. 104.)

536. Each of the values of the unknown quantity is called a *root of the equation.*

Thus, in the example above,

The roots of the quadratic equation are	3, 2,
of the cubic equation,	4, 3, 2,
of the biquadratic,	5, 4, 3, 2.

The term *root* is not to be understood in the same sense here, as in the preceding sections. The root of an *equation* is not a quantity which multiplied into *itself* will produce the equation. It is one of the values of the unknown quantity; and when its sign is changed by transposition, it is a term in one of the binomial factors which enter into the composition of the equation of which it is a root.

The value of the unknown letter x, in the equation, is a quantity which may be substituted for x, without affecting the equality of the members. In the equations which we are now considering, each member is equal to 0; and the first is the product of several factors. This product will continue to be equal to 0, as long as any one of its factors is 0. (Art. 106.) If then in the equation

$$(x-2)\times(x-3)\times(x-4)\times(x-5)=0,$$

we substitute 2 for x, in the first factor, we have

$$0\times(x-3)\times(x-4)\times(x-5)=0.$$

So, if we substitute 3 for x, in the second factor, or 4 in the third, or 5 in the fourth, the whole product will still be 0. This will also be the case, when the product is formed by an actual multiplication of the several factors into each other.

Thus, as $x^3-9x^2+26x-24=0$; (Art. 535.)

So $2^3-9\times2^2+26\times2-24=0$,

And $3^3-9\times3^2+26\times3-24=0$, &c.

Either of these values of x, therefore, will satisfy the conditions of the equation.

We have thus far been considering higher equations as formed, by multiplication, from simple equations. But the inquiry may arise, whether every equation of a higher degree can be regarded as the product of two or more simple equations. It is proposed, in the following articles, to answer this inquiry, and to bring into view a number of the most important general properties of equations.

537. An equation of the mth degree consists of x^m, the several inferior powers of x with their co-efficients, and one term in which x is not contained. If A, B, C, ... T, be put for the several co-efficients, and U for the last term, then

$$x^m+Ax^{m-1}+Bx^{m-2}+Cx^{m-3}\ldots.+Tx+U=0,$$

will be a general expression for an equation of any degree.

Any quantity which, substituted for x, will make the members equal, is called a *root* of the equation. (Arts. 335, 536.)

If (a) *is a root of the general equation of the* (m)*th degree, the first member is exactly divisible by* $(x-a)$.

For by substituting a for x, we have

$$a^m+Aa^{m-1}+Ba^{m-2}+Ca^{m-3}\ldots.+Ta+U=0.$$

And transposing terms,

$$U=-a^m-Aa^{m-1}-Ba^{m-2}-Ca^{m-3}\ldots.-Ta.$$

Substituting this value for U, in the original equation,

$$\left.\begin{array}{r} x^m+Ax^{m-1}+Bx^{m-2}+Cx^{m-3}\ldots.+Tx \\ -a^m-Aa^{m-1}-Ba^{m-2}-Ca^{m-3}\ldots.-Ta \end{array}\right\}=0.$$

Or, uniting the corresponding terms,

$$(x^m-a^m)+A(x^{m-1}-a^{m-1})+B(x^{m-2}-a^{m-2})+$$
$$C(x^{m-3}-a^{m-3})\ldots.+T(x-a)=0.$$

In this expression, each of the quantities (x^m-a^m) $A(x^{m-1}-a^{m-1})$, &c. is divisible by $x-a$; (Art. 130:) therefore the *whole* is divisible by $x-a$.

Conversely, If the first member of any equation be divisible by $x-a$, a is a root of the equation. For (Art. 114,) this member may be resolved into two factors, of which $x-a$ is one; and (Art. 106,) it must itself become zero, when $x=a$, because the factor $x-a$ becomes zero. The equation will therefore be satisfied, by giving to x the value a; and a must be one of its roots.

Ex. 1. Prove that 2 is a root of the equation

$$x^3-7x^2+12x-4=0.$$

This is to be done, by dividing by $x-2$. If there is no remainder, 2 must be a root of the equation.

Ex. 2. Prove that 3 is not a root of the equation

$$x^3+2x^2-6x-4=0.$$

When we divide here by $x-3$, we find a remainder; which shows that 3 is not a root.

Ex. 3. Find whether -1 is a root of the equation

$$x^4-x^3-5x^2+4x-3=0.$$

Ex. 4. Find whether 1 is a root of the equation

$$x^4-2x^3-11x^2-8x+15=0.$$

Ex. 5. Find whether -5 is a root of the equation

$$x^5+3x^3-64x^2+x+23=0.$$

538. *Every equation of the (m)th degree has exactly (m) roots.*

It will be assumed that every equation has at least one root.

Let a be a root of the equation

$$x^m+Ax^{m-1}+Bx^{m-2}\ldots.+Tx+U=0.$$

The first member is divisible by $(x-a)$ (Art. 537.) If we divide by this, the quotient will be a polynomial of the degree $(m-1)$; which may be written thus,

$$x^{m-1}+A'x^{m-2}+B'x^{m-3} \text{ \&c.}$$

If we make this equal to zero, and suppose b to be a root of the equation thus formed, the first member will be divisible by $x-b$; and if we divide by this, the result will be a polynomial of the degree $(m-2)$. By proceeding in this way till we have divided $(m-1)$ times, we shall obtain a

simple equation, having only one root, which may be denoted by l. Hence the original equation has m roots, $a, b, c, \ldots l$; and the first member of it is composed of the m factors, $x-a,\ x-b,\ x-c, \ldots x-l$.

The equation can have no other root; for none of the factors $x-a$, $x-b$, &c. of which the first member is composed, can be zero, unless x equals one of the quantities, a, b, &c.

Ex. 1. One root of the equation

$$x^3-4x^2+2x+3=0$$

is 3. What are the other roots?

If we divide by $x-3$, we obtain the equation $x^2-x-1=0$; which must contain the two required roots. By solving this quadratic, the roots will be found to be $\frac{1\pm\sqrt{5}}{2}$.

Ex. 2. One root of the equation

$$x^3-5x^2+3x+9=0$$

is 3. What are the other two? Ans. 3 and -1.

Ex. 3. One root of the equation

$$x^3+5x^2-3x-7=0$$

is -1. What are the other roots? Ans. $-2\pm\sqrt{11}$.

Ex. 4. Two roots of the equation

$$x^4-x^3+2x^2-14x+12=0$$

are 1 and 2. What are the other roots? Ans. $-1\pm\sqrt{-5}$.

Ex. 5. Two roots of the equation

$$x^4+x^3-7x^2-x+6=0$$

are 1 and -3. What are the other roots?

Ex. 6. One root of the equation

$$x^4-2x^3+x^2-5x-9=0$$

is -1. Find the equation containing the other roots.

539. The roots of an equation are not always real. Some or even all of them may be imaginary. In the fourth example above, two of the roots are imaginary, and two real.

It often happens that the roots of an equation are not all unequal. Thus, in the second of the preceding examples, the roots are 3, 3, and -1; two of which are alike.

540. The laws by which the *co-efficients* of an equation are governed, may be seen, from the following view of the multiplication of the factors

$$x-a,\ x-b,\ x-c,\ x-d,$$

each of which is supposed equal to 0.

The several co-efficients of the same power of x, are placed *under* each other.

Thus, $-ax-bx$ is written $\left.\begin{matrix}-a\\-b\end{matrix}\right\}x$; and the other co-efficients in the same manner.

The product, then

$$\begin{array}{ll}\text{Of} & (x-a)=0\\ \text{Into} & \underline{(x-b)=0}\end{array}$$

$$\text{Is } x^2\left.\begin{matrix}-a\\-b\end{matrix}\right\}x+ab=0, \text{ a quadratic equation.}$$

This into $x-c=0$

$$\text{Is } x^3\left.\begin{matrix}-a\\-b\\-c\end{matrix}\right\}x^2\left.\begin{matrix}+ab\\+ac\\+bc\end{matrix}\right\}x-abc=0, \text{ a cubic equation.}$$

This into $x-d=0$

$$\text{Is } x^4\left.\begin{matrix}-a\\-b\\-c\\-d\end{matrix}\right\}x^3\left.\begin{matrix}+ab\\+ac\\+ad\\+bc\\+bd\\+cd\end{matrix}\right\}x^2\left.\begin{matrix}-abc\\-abd\\-acd\\-bcd\end{matrix}\right\}x+abcd=0, \text{ a biquadratic.}$$

&c.

541. By attending to these equations, it will be seen that,

In the *first* term of each, the co-efficient of x is 1;

In the *second* term, the co-efficient is the sum of all the roots of the equation, with contrary signs. Thus the roots of the quadratic equation are a and b, and the co-efficients, in the second term, are $-a$ and $-b$.

In the *third* term, the co-efficient of x, is the sum of all the products which can be made, by multiplying together any *two* of the roots. Thus, in the cubic equation, as the roots are a, b, and c, the co-efficients, in the third term, are ab, ac, bc.

In the *fourth* term, the co-efficient of x is the sum of all the products which can be made, by multiplying together any

three of the roots after their signs are changed. Thus the roots of the biquadratic equation are a, b, c and d, and the co-efficients in the fourth term are $-abc$, $-abd$, $-acd$, $-bcd$.

The *last term* is the product formed from *all* the roots of the equation after the signs are changed.

In the cubic equation, it is $-a\times-b\times-c=-abc$.

In the biquadratic, $-a\times-b\times-c\times-d=+abcd$, &c.

542. In Art. 540, the roots of the equations are all represented as *positive*. The signs are changed by transposition, and when the several factors are multiplied together, the terms in the product, as in the power of a residual quantity, (Art. 495,) are alternately positive and negative. But if the roots are all *negative*, they become positive by transposition, and *all* the terms in the product must be positive. Thus if the several values of x are $-a$, $-b$, $-c$, $-d$, then

$$x+a=0,\quad x+b=0,\quad x+c=0,\quad x+d=0;$$

and by multiplying these together, we shall obtain the same equations as before, except that the signs of all the terms will be positive. In other cases, some of the roots may be positive, and some of them negative.

Ex. 1. Form the equation whose roots are 1, -1, 2, -3.
Ans. $x^4+x^3-7x^2-x+6=0$.

This result may be obtained, either by multiplying together the factors $x-1$, $x+1$, $x-2$, $x+3$, or by ascertaining the co-efficients according to the law in Art. 541.

Ex. 2. Form the equation whose roots are 1, 2, -2, 3; both by multiplying together its factors $x-1$, $x-2$, &c. and by obtaining the co-efficients from the law in Art. 541.
Ans. $x^4-4x^3-x^2+16x-12=0$.

Ex. 3. Find the equation whose roots are -1, 2, -3.
Ans. $x^3+2x^2-5x-6=0$.

Ex. 4. Find the equation whose roots are 1, 2, 2, -2, -3.
Ans. $x^5-11x^3+6x^2+28x-24=0$.

543. The *quotient* produced by dividing the original equation in Art. 537, by $x-a$, is evidently equal to the aggregate of the particular quotients arising from the division of the several quantities (x^m-a^m), $A(x^{m-1}-a^{m-1})$, &c.

The quotient of $(x^m-a^m)\div(x-a)$, (Art. 130.) is

$$x^{m-1}+ax^{m-2}+a^2x^{m-3}+a^3x^{m-4}\ldots+a^{m-1}.$$

The quotient of $A(x^{m-1}-a^{m-1})\div(x-a)$ is

$$Ax^{m-2}+Aax^{m-3}+Aa^2x^{m-4}\ldots+Aa^{m-2}.$$

&c. &c.

Collecting these particular quotients together, and placing under each other the co-efficients of the same power of x, we have the following expression for the quotient of

$$x^m+Ax^{m-1}+Bx^{m-2}+Cx^{m-3}\ldots\ldots+Tx+U$$

divided by $x-a$.

$$\text{I.}\qquad x^{m-1}\left.\begin{array}{r}+a\\+A\end{array}\right\}x^{m-2}\left.\begin{array}{r}+a^2\\+Aa\\+B\end{array}\right\}x^{m-3}\left.\begin{array}{r}+a^3\\+Aa^2\\+Ba\\+C\end{array}\right\}x^{m-4}\ldots\begin{array}{l}+a^{m-1}\\+Aa^{m-2}\\+Ba^{m-3}\\+Ca^{m-4}\\\ldots\ldots\\+T.\end{array}$$

And if b is a root of the equation, the quotient from dividing by $x-b$, is

$$\text{II.}\qquad x^{m-1}\left.\begin{array}{r}+b\\+A\end{array}\right\}x^{m-2}\left.\begin{array}{r}+b^2\\+Ab\\+B\end{array}\right\}x^{m-3}\left.\begin{array}{r}+b^3\\+Ab^2\\+Bb\\+C\end{array}\right\}x^{m-4}\ldots\begin{array}{l}+b^{m-1}\\+Ab^{m-2}\\+Bb^{m-3}\\+Cb^{m-4}\\\ldots\ldots\\+T.\end{array}$$

And if c is another root, the quotient from dividing by $x-c$, is

$$\text{III.}\qquad x^{m-1}\left.\begin{array}{r}+c\\+A\end{array}\right\}x^{m-2}\left.\begin{array}{r}+c^2\\+Ac\\+B\end{array}\right\}x^{m-3}\left.\begin{array}{r}+c^3\\+Ac^2\\+Bc\\+C\end{array}\right\}x^{m-4}\ldots\begin{array}{l}+c^{m-1}\\+Ac^{m-2}\\+Bc^{m-3}\\+Cc^{m-4}\\\ldots\\+T.\end{array}$$

In the same manner may be found the quotients produced by introducing successively into the divisor the several roots of the equation; which are equal in number to m.

544. From the known relations between the roots and the co-efficients of equations, as stated in Art. 541, Newton has derived a method of determining the co-efficients, from the *sum* of the roots, the sum of their *squares*, the sum of their *cubes*, &c., though the roots themselves are unknown; and on the other hand of determining from the co-efficients, the sum of the roots, the sum of their squares, the sum of their cubes, &c. For this purpose, the following plan of notation is adopted. S_1 is put for the sum of the roots, S_2 for

the sum of their *squares*, S_3 for the sum of their *cubes*, &c. If the roots are $a, b, c, d, \ldots l$, then

$$\begin{aligned}
S_1 &= a+b+c+d\ldots+l \\
S_2 &= a^2+b^2+c^2+d^2\ldots+l^2 \\
S_3 &= a^3+b^3+c^3+d^3\ldots+l^3 \\
S_m &= a^m+b^m+c^m+d^m\ldots+l^m \\
S_{m-1} &= a^{m-1}+b^{m-1}+c^{m-1}+d^{m-1}\ldots+l^{m-1} \\
&\text{&c.} \qquad\qquad\qquad \text{&c.}
\end{aligned}$$

By means of this notation, we obtain the following expression for the *sum* of all the quotients marked I, II, III, &c. (Art. 543,) and continued till their number is equal to m.

$$\begin{matrix} mx^{m-1}+S_1 \\ +mA \end{matrix}\Bigg\} x^{m-2} \begin{matrix} +S_2 \\ +AS_1 \\ +mB \end{matrix}\Bigg\} x^{m-3} \begin{matrix} +S_3 \\ +AS_2 \\ +BS_1 \\ +mC \end{matrix}\Bigg\} x^{m-4} \quad \begin{matrix} \ldots+S_{m-1} \\ +AS_{m-2} \\ +BS_{m-3} \\ +CS_{m-4} \\ \ldots\ldots \\ +mT. \end{matrix}$$

$Y.$

In the original equation,

$$x^m+Ax^{m-1}+Bx^{m-2}+Cx^{m-3}\ldots+Tx+U=0,$$

the co-efficients, A, B, C, &c. have determinate relations to the sum and products of the roots, a, b, c, &c. (Art. 541.) But the quotient marked I, (Art. 543,) produced by dividing by $x-a$, is the first member of an equation of the *next inferior degree*, from which the root a is *excluded*, since it is the product of $x-b$, $x-c, \ldots x-l$, (Art. 538.) So b is excluded from the quotient II, c from the quotient III, &c. In the expression above marked Y, which is the *sum* of m quotients, the co-efficient of x in the second term is S_1+mA. But A, which is the co-efficient of x in the second term of the original equation, is equal to the sum of the roots a, b, c, &c. with contrary signs; (Art. 541,) that is, $S_1=-A$.

Therefore,

$$S_1+mA=(m-1)A.$$

In the *third* term of the original equation, B the co-efficient of x, is equal to the sum of all the products which can be made by multiplying together any *two* of the roots. (Art. 541.) But each of these products will be excluded from *two* of the quotients, I, II, III, &c. For instance, ab will not be found in the first, from which a is excluded, nor in the second, from which b is excluded. Therefore in the expression Y,

the co-efficient of x in the third term is equal to $mB-2ab-2ac-2ad$, &c. But $-2ab, -2ac, -2ad$, &c. $=-2B$. So that

$$S_2+AS_1+mB=(m-2)B.$$

In the *fourth* term of the original equation, C the co-efficient of x, is equal to the sum of all the products which can be made by multiplying together any *three* of the roots, after their signs are changed. But each of these products will be excluded from *three* of the quotients, I, II, III, &c. So that, in the expression Y, the co-efficient of x in the fourth term, is equal to $mC-3abc-3abd$, &c. That is,

$$S_3+AS_2+BS_1+mC=(m-3)C.$$

In the same manner, the values of the co-efficients of x in succeeding terms may be found; the number of the co-efficients being one less than the number of roots in the equation.

Collecting these results, we have

$$\begin{aligned} S_1+mA&=(m-1)A,\\ S_2+AS_1+mB&=(m-2)B,\\ S_3+AS_2+BS_1+mC&=(m-3)C,\\ S_4+AS_3+BS_2+CS_1+mD&=(m-4)D,\\ \text{\&c.}\qquad & \qquad\text{\&c.} \end{aligned}$$

Transposing and uniting terms,

I
$$\begin{aligned} &S_1+A=0,\\ &S_2+AS_1+2B=0,\\ &S_3+AS_2+BS_1+3C=0,\\ &S_4+AS_3+BS_2+CS_1+4D=0,\\ &\text{\&c.}\qquad\qquad\text{\&c.} \end{aligned}$$

Substituting for S_1, S_2, S_3, &c. their values, and reducing,

II.
$$\begin{aligned} S_1&=-A,\\ S_2&=\ \ A^2-2B,\\ S_3&=-A^3+3AB-3C,\\ S_4&=\ \ A^4-4A^2B+4AC+2B^2-4D,\\ &\text{\&c.}\qquad\qquad\text{\&c.} \end{aligned}$$

We have here obtained symmetrical expressions for the sum of the roots of an equation, the sum of their squares, the sum of their cubes, &c. in terms of the co-efficients. By transposing the terms in the expressions marked I, we have the following values of A, B, C, &c.

III.
$$\begin{aligned}
A&=-S_1\\
B&=-\tfrac{1}{2}(AS_1+S_2)\\
C&=-\tfrac{1}{3}(BS_1+AS_2+S_3)\\
D&=-\tfrac{1}{4}(CS_1+BS_2+AS_3+S_4)\\
&\&c. \qquad\qquad \&c.
\end{aligned}$$

By which the *co-efficients* of an equation may be found from the sum of its roots, the sum of their squares, the sum of their cubes, &c.

Ex. 1. Required the sum of the roots, the sum of their squares, and the sum of their cubes, in the equation

$$x^4-10x^3+35x^2-50x-24=0.$$

Here $A=-10.$ $\quad B=35.$ $\quad C=-50.$

Therefore $S_1=10$

$$S_2=10^2-(2\times35)=30$$

$$S_3=10^3+(3\times-10\times35)-(3\times-50)=100.$$

2. Required the terms of the biquadratic equation in which $S_1=1$, $S_2=39$, $S_3=-89$, and the *product* of all the roots after their signs are changed is -30.

Ans. $x^4-x^3-19x^2+49x-30=0.$

545. There is occasion in some of the following investigations to substitute the sum of two quantities for a single quantity in a given polynomial. When the terms of the result are arranged according to the powers of one of the new quantities, the co-efficients of this quantity are found to follow a remarkable law; and may be readily derived from each other.

In the polynomial

$$x^m+Ax^{m-1}+Bx^{m-2} \;\&c.$$

let $r+y$ be put for x; it will then be

$$(r+y)^m+A(r+y)^{m-1}+B(r+y)^{m-2} \;\&c.$$

And if we develope the powers of $r+y$ by the binomial theorem, and arrange the resulting terms according to the powers of y, we have

$$r^m+Ar^{m-1}+Br^{m-2} \;\&c.$$

$$+(mr^{m-1}+(m-1)Ar^{m-2}+(m-2)Br^{m-3} \;\&c.)\,y$$

$$+\left(\frac{m(m-1)}{2}r^{m-2}+\frac{(m-1)(m-2)}{2}Ar^{m-3}+\frac{(m-2)(m-3)}{2}Br^{m-4} \;\&c.\right)y^2$$

$$+\left(\frac{m(m-1)(m-2)}{2.3}r^{m-3}+\frac{(m-1)(m-2)(m-3)}{2.3}Ar^{m-4}+\right.$$
$$\left.\frac{(m-2)(m-3)(m-4)}{2.3}Br^{m-5}\ \&c.\right)y^3+\&c.$$

For the sake of brevity, we will put

$R=r^m+Ar^{m-1}+Br^{m-2}$ &c.

$R'=mr^{m-1}+(m-1)Ar^{m-2}+(m-2)Br^{m-3}$ &c.

$R''=m(m-1)r^{m-2}+(m-1)(m-2)Ar^{m-3}+(m-2)(m-3)Br^{m-4}$ &c.

$R'''=m(m-1)(m-2)r^{m-3}+(m-1)(m-2)(m-3)Ar^{m-4}$
$+(m-2)(m-3)(m-4)Br^{m-5}$ &c.
&c. &c.

The preceding polynomial then becomes

$$R+R'y+\frac{R''}{2}y^2+\frac{R'''}{2.3}y^3\ \&c.$$

And it may be seen that

R is derived from the given polynomial, by substituting r for x;

R' is derived from R by multiplying each term by the exponent of r in that term, and diminishing the exponent by unity;

And each of the polynomials R'', R''', &c. is derived from the preceding one, in the same manner as R' from R.

R' is called the *first derived* polynomial of R,
R'' the second derived polynomial of R,
or the first derived polynomial of R',
and so on.

Ex. 1. Find the successive derived polynomials of r^3-r^2+2r+6. Ans. $3r^2-2r+2$, $6r-2$, 6.

Ex. 2. Find the derived polynomials of z^4+2z^2-6z-4.
Ans. $4z^3+4z-6$, $12z^2+4$, $24z$, 24.

Ex. 3. Find the derived polynomials of $x^5-3x^4-3x^2+5$.
Ans. $5x^4-12x^3-6x$, $20x^3-36x^2-6$, $60x^2-72x$, $120x-72$, 120.

Ex. 4. What are the derived polynomials of
$$x^4-x^3+2x^2-3x+4?$$

546. *To transform an equation into another whose roots shall be greater or less than those of the former, by any given quantity.*

Let the given equation be

$$x^m + Ax^{m-1} + Bx^{m-2} \text{ \&c.} = 0;$$

which it is required to transform into another, whose roots are *less* than its own, by r. Put $y=x-r$. Then $x=r+y$; and by substitution, the given equation will become

$$R + R'y + R''\frac{y^2}{2} + R'''\frac{y^3}{2.3} \text{ \&c.} = 0;$$

the polynomials R, R', R'', &c. being determined by the rule in the preceding article. Since $y=x-r$, the values of y which will satisfy the last equation, are less by r than those of x in the former equation; that is, the roots of the new equation, in which y is the unknown quantity, are less by r than those of the original equation in which the unknown quantity is x.

To render the roots of the new equation *greater* by r than those of the original equation, we have only to change the sign of r, putting $y=x+r$.

Ex. 1. Transform the equation

$$x^3 - 7x^2 - 11x + 42 = 0$$

into another, whose roots are less than those of the first by r.

Put $y=x-r$, then by Art. 545,

$$\begin{aligned} R &= r^3 - 7r^2 - 11r + 42 \\ R' &= 3r^2 - 14r - 11 \\ R'' &= 6r - 14 \\ R''' &= 6. \end{aligned}$$

Hence,

$$(r^3 - 7r^2 - 11r + 42) + (3r^2 - 14r - 11)y + \frac{6r-14}{2}y^2 + \frac{6}{2.3}y^3 = 0,$$

or writing this in the usual way,

$$y^3 + (3r-7)y^2 + (3r^2 - 14r - 11)y + r^3 - 7r^2 - 11r + 42 = 0.$$

Ex. 2. Transform

$$x^3 + x^2 - 14x + 12 = 0,$$

into an equation whose roots are *greater* than those of the former by r.

Put $y=x+r$. The derived polynomials, if r had the contrary sign, would be

$$\begin{aligned} R &= r^3 + r^2 - 14r + 12 \\ R' &= 3r^2 + 2r - 14 \\ R'' &= 6r + 2 \\ R''' &= 6. \end{aligned}$$

Then by changing the signs of the odd powers of each, and remembering that R'' is to be divided by 2, and R''' by 2.3, or 6, we shall obtain the co-efficients of y in the required equation; which accordingly is

$$y^3-(3r-1)y^2+(3r^2-2r-14)y-r^3+r^2+14r+12=0.$$

Ex. 3. Find the equation whose roots are greater by 2 than those of the equation

$$x^4-2x^3+5x^2+4x-8=0.$$

Ans. $y^4-10y^3+41y^2-72y+36=0.$

Ex. 4. Find the equation whose roots are less by 1 than those of the equation

$$x^5+3x^3-7x^2+x+12=0.$$

Ans. $y^5+5y^4+13y^3+12y^2+y+10=0.$

Ex. 5. What is the equation whose roots are greater by 1 than those of the equation

$$x^4-5x^2-6x-2=0?$$

Ans. $y^4-4y^3+y^2=0.$

One root of the last equation is obviously zero. And if we divide by $y-0$, that is by y, the result is $y^3-4y^2+y=0$; which also has zero for one of its roots. Dividing again by y, we obtain the quadratic equation $y^2-4y+1=0$; whose roots are $2\pm\sqrt{3}$. From this we see that the roots of the original equation must be -1, -1, $1+\sqrt{3}$, $1-\sqrt{3}$.

547. There is another method of transforming equations, so as to increase or diminish their roots, which is often more convenient than the preceding. In employing it, we have occasion to perform a number of successive divisions. They are all, however, simple and of a similar kind. Before proceeding to explain the method, we will show how these divisions are most conveniently performed. In each of them, the divisor is a binomial of the form $x-a$, and the dividend a polynomial, as $Ax+B$, Ax^2+Bx+C, Ax^3+Bx^2+Cx+D, or one of some higher degree with respect to x. Let the proposed dividend be

$$Ax^4+Bx^3+Cx^2+Dx+E,$$

If we divide this by $x-a$, the first term of the quotient will be Ax^3. Let the whole quotient be represented by $Ax^3+B'x^2+C'x+D'$, and the remainder by E'. Then, as

the product of the divisor and quotient, together with the remainder, is equal to the dividend, we have the equation

$$Ax^4+Bx^3+Cx^2+Dx+E=(x-a)(Ax^3+B'x^2+C'x+D')+E'$$
$$=Ax^4+(B'-aA)x^3+(C'-aB')x^2+(D'-aC')x+E'-aD'$$

Hence,

$$\begin{array}{lll} B'-aA=B & \text{and} & B'=B+aA \\ C'-aB'=C & & C'=C+aB' \\ D'-aC'=D & & D'=D+aC' \\ E'-aD'=E & & E'=E+aD' \end{array}$$

By these last formulas, it is easy to compute successively the unknown co-efficients of the quotient, namely, B', C', D', and the remainder E'.

The same method may be pursued, whatever be the degree of the dividend. Hence,

To divide a polynomial of the form

$$Ax^m+Bx^{m-1}+Cx^{m-2}\ldots+Tx+U,$$

by the binomial $x-a$,

Represent the quotient by

$$Ax^{m-1}+B'x^{m-2}+C'x^{m-3}\ldots+T',$$

and the remainder by U'.

Then find the co-efficient

$$\begin{array}{lll} B' & \textit{by the equation} & B'=B+aA \\ C' & \textit{by the equation} & C'=C+aB' \\ & \&c. & \&c. \end{array}$$

the final equation being $\quad U'=U+aT'$.

To divide by $x+a$ instead of $x-a$, we have only to change the sign of a in the preceding formulas, making

$$B'=B-aA$$
$$C'=C-aB'$$
&c.

Ex. 1. Divide $2x^4+7x^3-8x^2-11x+18$ by $x-3$.

Ans. $2x^3+13x^2+31x+82+\dfrac{264}{x-3}$.

The process for obtaining this, according to the rule, is as follows.

A	B	C	D	E
2	7	-8	-11	18
	6	39	93	246
	13	31	82	264
	B'	C'	D'	E'

Here 2 is multiplied by 3 and the product added to 7; the sum 13 or B' is multiplied by 3 and the product added to -8; and so on.

Ex. 2. Divide $x^5-4x^4+9x^3+12x^2-10x-2$ by $x-2$.

The required operation is as follows.

1	-4	9	12	-10	-2
	2	-4	10	44	68
	-2	5	22	34	66

$$\text{Ans. } x^4-2x^3+5x^2+22x+34+\frac{66}{x-2}.$$

The work, here and in other cases, may be shortened by omitting to write the second line of numbers. Thus, in the preceding case, when 1 is multiplied by 2 and the product added to -4, we may omit to write the product 2, and set down only the sum -2. In like manner we may omit the product -4, and write only the sum 5; and so on. The whole operation will then be presented in this abbreviated form,

1	-4	9	12	-10	-2
	-2	5	22	34	66

Ex. 3. Divide $2x^4-5x^2-14x-20$ by $x-1$.

$$\text{Ans. } 2x^3+2x^2-3x-17-\frac{37}{x-1}.$$

When, as in this example, any term of the dividend is wanting, the place of its co-efficient must be supplied with zero.

Ex. 4. Divide $5x^3-5x^2+10x-12$ by $x+2$.

$$\text{Ans. } 5x^2-15x+40-\frac{92}{x+2}.$$

Ex. 5. Divide $x^5-5x^4-6x^3-x+1$ by $x+1$.

$$\text{Ans. } x^4-6x^3-1+\frac{2}{x+1}.$$

548. We will now explain the method of transforming equations, *by successive divisions*, in order to increase or diminish their roots.

Let it be required to transform

$$Ax^4+Bx^3+Cx^2+Dx+E=0$$

into another equation whose roots are *less* than those of the former by a. If we take $y=x-a$, that is, $x=y+a$, and substitute this value for x in the above equation, we shall obtain the equation required. This equation will be of the form

$$Ay^4+B'y^3+C'y^2+D'y+E'=0.$$

In order to determine the co-efficients B', C', D' and E' more conveniently than by the substitution just mentioned, we observe that if $x-a$ be put for y, the last equation will be converted into the original equation; that is,

$$Ax^4+Bx^3+Cx^2+Dx+E$$
$$=A(x-a)^4+B'(x-a)^3+C'(x-a)^2+D'(x-a)+E'$$

must be an identical equation (Art. 173.) Now if we divide the second member by $x-a$, the remainder will be E', and the quotient

$$A(x-a)^3+B'(x-a)^2+C'(x-a)+D';$$

and if we divide the latter by $x-a$, the remainder will be D', and so on; the unknown quantities E', D', C', B' being the remainders of the successive divisions. Then if the first member be divided in the same way by $x-a$, the remainders of the successive divisions must be the same, namely, E', D', C', B'; and thus these quantities may be easily determined.

In a similar manner, the co-efficients of the required equation may be found, whatever be the degree of the given equation. Hence,

To transform an equation into another whose roots are *less* than those of the former by a given quantity a, we have, in addition to the method in Art. 546, the following rule.

Divide the first member of the equation by $x-a$, (x being the unknown quantity;) reserve the remainder and divide the quotient by $x-a$; reserve the new remainder and divide the new quotient by $x-a$; and so on. The first remainder will be the last term of the required equation; the second remainder will be the co-efficient of y, the unknown quantity; the third remainder will be the co-efficient of y^2; &c.

If the roots of the given equation are to be *increased* by a, we must divide by $x+a$, instead of $x-a$.

The divisions are to be made by the rule in the preceding article.

Ex. 1. Find the equation whose roots are less by 2 than those of the equation

$$2x^4-5x^3+3x^2-x-4=0.$$

The required divisions may be performed as follows,

2	-5	3	-1	-4
	4	-2	2	2
	-1	1	1	-2
	4	6	14	
	3	7	15	
	4	14		
	7	21		
	4			
	11			

The divisor here being $x-2$, the co-efficients of the first quotient, obtained as in Art. 547, are 2, -1, 1, 1, and the remainder -2. All except the first of these are in the third line of numbers above; the first co-efficient 2, being the same as that of the original equation, is at the beginning of the first line. The remainder -2 being reserved, and the quotient, (whose co-efficients have been found to be 2, -1, 1, 1,) being divided by $x-2$, (Art. 547,) the co-efficients of the new equation are found to be 2, 3, 7, and the remainder 15. Another division gives 2 and 7 for the co-efficients of the quotient, and 21 for the remainder. And the division ends by giving the quotient 2 and the remainder 11. The required equation therefore is

$$2y^4+11y^3+21y^2+15y-2=0.$$

Ex. 2. Find the equation whose roots are less by the decimal .1 than those of the equation

$$2y^4+11y^3+21y^2+15y-2=0.$$

The required operation is as follows.

2	11	21	15	-2
	.2	1.12	2.212	1.7212
	11.2	22.12	17.212	$-$ .2788
	.2	1.14	2.326	
	11.4	23.26	19.538	
	.2	1.16		
	11.6	24.42		
	.2			
	11.8			

or more briefly, (Art. 547, Ex. 2,)

2	11	21	15	−2
	11.2	22.12	17.212	− .2788
	11.4	23.26	19.538	
	11.6	24.42		
	11.8			

Ans. $2z^4+11.8z^3+24.42z^2+19.538z-.2788=0.$

Ex. 3. *Increase* by 3 the roots of the equation

$$x^5+2x^3+3x^2-4x-5=0.$$

Here the co-efficients are 1, 0, 2, 3, −4, −5. We then proceed as follows,

1	0	2	3	−4	−5
	−3	9	−33	90	−258
	−3	11	−30	86	−263
	−3	18	−87	351	
	−6	29	−117	437	
	−3	27	−168		
	−9	56	−285		
	−3	36			
	−12	92			
	−3				
	−15				

Or

1	0	2	3	−4	−5
	−3	11	−30	86	−263
	−6	29	−117	437	
	−9	56	−285		
	−12	92			
	−15				

Ans. $y^5-15y^4+92y^3-285y^2+437y-263=0.$

Ex. 4. Diminish by .02 the roots of the equation

$$2x^4-x^3-10x+12=0.$$

Ans. $2y^4-.84y^3-.0552y^2-10.001136y+11.79999232=0.$

Ex. 5. Diminish by .03 the roots of the equation

$$4x^3-.36x^2+.08x+2.15=0.$$

Ans. $4y^3+.0692y+2.152184=0.$

549. It is sometimes required *to transform an equation into another whose second term is wanting.*

Let the given equation be

$$x^m+Ax^{m-1}+Bx^{m-2} \text{ \&c.} =0.$$

If in this we substitute $y+r$ for x, and develope the powers of $y+r$ by the binomial theorem, we see that no power of y as high as y^{m-1} can result, except from the first two terms, namely, x^m and Ax^{m-1}, that is, $(y+r)^m$ and $A(y+r)^{m-1}$. The first of these developed is y^m+mry^{m-1} &c. The second is $Ay^{m-1}+(m-1)Ary^{m-2}$ &c. Taking their sum, we find the term containing y^{m-1} to be $(mr+A)y^{m-1}$. In order to make this disappear, so that the new equation, having y for its unknown quantity, may be of the form

$$y^m+B'y^{m-2}+C'y^{m-3} \text{ \&c.} =0,$$

we must suppose that $mr+A=0$; in which case $r=-\frac{A}{m}$, and $y+r$, that is x, $=y-\frac{A}{m}$.

Hence, to make the second term of an equation disappear,

Substitute for the unknown quantity, a new unknown quantity, together with the co-efficient of the second term, taken with a contrary sign, and divided by the number answering to the degree of the equation.

Ex. 1. Transform $x^3-6x^2+7=0$ into an equation whose second term is wanting. Ans. $y^3-12y-9=0$.

Here we substitute for x the new unknown quantity y, together with $+2$, which is a *third* of the co-efficient -6, taken with the contrary sign.

Ex. 2. Transform $x^3+3x^2-3x-1=0$, into an equation whose second term is wanting. Ans. $y^3-6y+4=0$.

Ex. 3. Transform $x^4-8x^3-5x+12=0$ into an equation whose second term is wanting.

Ans. $y^4-24y^2-69y-46=0$.

Ex. 4. Transform $x^5+20x^4+2x^2-3x-24=0$ into an equation whose second term is wanting.

550. *If the signs of the alternate terms of an equation be changed, the signs of all the roots will be changed.*

Let a be one of the roots of the equation

$$x^m+Ax^{m-1}+Bx^{m-2}+Cx^{m-3} \text{ \&c.} =0;$$

then $-a$ is a root of the equation

$$x^m-Ax^{m-1}+Bx^{m-2}-Cx^{m-3} \text{ \&c.} =0;$$

that is, $-a$, when put for x will reduce the first member to zero. For when $-a$ is substituted for x, the first member becomes

$$a^m+Aa^{m-1}+Ba^{m-2}+Ca^{m-3} \text{ \&c.}$$

if m is an even number; and it becomes

$$-a^m-Aa^{m-1}-Ba^{m-2}-Ca^{m-3} \text{ \&c.}$$

that is, $\quad -(a^m+Aa^{m-1}+Ba^{m-2}+Ca^{m-3} \text{ \&c.}),$

if m is odd. The value in either case must be zero. For since a is a root of the equation first proposed,

$$a^m+Aa^{m-1}+Ba^{m-2}+Ca^{m-3} \text{ \&c.} =0.$$

In this demonstration, it is implied that the equation is complete; or at least that its terms contain alternately odd and even powers of x. When this is not the case, we should supply the place of every term that is wanting, with zero; so as to render the equation complete in form, before applying to it the preceding proposition.

Ex. 1. The roots of the equation

$$x^3+4x^2+x-6=0,$$

are 1, 2, -3. What are those of the equation

$$x^3-4x^2+x+6=0\,?$$

Ex. 2. The roots of the equation

$$x^4-4x^3-7x^2+34x-24=0,$$

are 1, 2, -3, 4. What are those of the equation

$$x^4+4x^3-7x^2-34x-24=0\,?$$

Ex. 3. The roots of the equation

$$x^4+4x^3+3x^2-4x-4=0,$$

are 1, -1, -2, -2. What are those of the equation

$$x^4-4x^3+3x^2+4x-4=0\,?$$

Ex. 4. The roots of the equation

$$x^3-7x+6=0$$

are 1, 2, and -3. What is the equation whose roots are -1 -2 and 3?

551. Through the remainder of this section, with the exception of Arts. 561 —— 563, it is to be understood that the co-efficients $A, B, C, \ldots T, U,$ occurring in equations or polynomials, are all *real* quantities.

552. *In any polynomial of the form*

$$x^m+Ax^{m-1}+Bx^{m-2}\ldots+Tx+U,$$

a value can be given to (x) *sufficiently large to render the first term greater than the sum of all the rest.*

To take the most unfavorable case, suppose A, B,... U to be all positive, or all negative. If L be put for the largest of these quantities, the sum of all the terms after the first will be less than

$$Lx^{m-1}+Lx^{m-2}\ldots+Lx+L,$$

that is, $L(x^{m-1}+x^{m-2}\ldots+x+1)$ or (Art. 452,) $L\times\dfrac{x^m-1}{x-1}$.

Now in order that this quantity be less than x^m, it is sufficient that x be equal to $L+1$. For $\dfrac{L}{x-1}$, will then be equal to unity, and $L\times\dfrac{x^m-1}{x-1}$ equal to x^m-1; which is less than x^m. Hence the sum of the terms

$$Ax^{m-1}+Bx^{m-1}\ldots+Tx+U$$

must also be less than x^m.

Any value of x greater than $L+1$, will likewise render the first term of the given polynomial greater than the sum of all the rest.

Ex. 1. What value of x will render the first term of the polynomial

$$x^4+9x^3+7x^2-10x-8$$

greater than the sum of all the rest?

Here the largest co-efficient is 10. Increasing this by 1, we have 11 for the required value of x.

Ex. 2. What value of x will make the first term of the polynomial

$$x^3-11x^2-10x-9$$

greater than the sum of the other terms?

Ex. 3. What value of x will make the first term of the polynomial

$$x^4+7x^3+6x^2-7x+5$$

greater than the sum of the others?

The above theorem will be employed in the proof of some of the following propositions concerning the roots of equations.

553. *If two numbers, when substituted for the unknown quantity in the first member of an equation, give results of different signs, the equation has at least one root comprised between those numbers.*

Let a certain number p, when put for x in the equation

$$x^m+Ax^{m-1}+Bx^{m-2} \text{ \&c. } =0,$$

make the first member positive; and another number q render it negative: the equation must have at least one root between p and q.

By supposing x to vary in value uninterruptedly, the first member of the equation will be made to vary uninterruptedly. But x in varying from p to q, makes the first member of the equation pass from a positive to a negative value. Now a quantity, varying uninterruptedly, can not change its value from positive to negative, without first becoming zero. There must therefore be some value of x between p and q, which satisfies the proposed equation, making its first member equal to zero; that is, there must be at least one root of the equation between the numbers p and q.

If the numbers p and q differ only by unity, the smaller number is the integral part of the root comprized between the two.

Ex. 1. What is the integral part of a root of the equation

$$x^3-7x^2+3x-4=0\,?$$

It will be found, by trial, that the contiguous numbers 6 and 7, when substituted for x in the first member of the equation give results with contrary signs. Hence the equation has a root comprized between 6 and 7. The integral part is therefore 6.

Ex. 2. Find the integral parts of two roots of the equation

$$x^4-12x^3-20x^2-36=0.$$

Ans. 13 and -2.

Ex. 3. Find the integral parts of two roots of the equation

$$x^4-4x^3+5x^2-2x-6=0.$$

Ans. 2 and 0.

Ex. 4. What is the integral part of a root of the equation

$$x^3-x^2+7=0\,?$$

554. *Every equation of an odd degree has at least one real root, of a different sign from that of its last term.*

Let the equation be

$$x^m+Ax^{m-1}+Bx^{m-2}\ldots\pm U=0.$$

And first, let the last term be *positive.* If zero be put for x in the first member of the equation, the result will be positive; being $+U$. But such a value can be given to x, that x^m shall exceed the sum of all the other terms. (Art. 552.) Let this value, which we will denote by p, be taken negatively. Then x^m being an odd power of x, will be negative; and being greater than the sum of all the following terms, the first member of the equation must be negative. Since then zero and $-p$, when put for x in the first member of the equation, give results with different signs, the equation has a *negative* root between 0 and $-p$. (Art. 553.)

Again, let the last term be *negative.* Then if zero be put for x, the first member of the equation becomes $-U$, a negative result. And by giving to x a positive value sufficiently large, which we will denote by p', the first member of the equation may be made positive. There must then be some *positive* number between 0 and p', which is a root of the equation.

555. *Every equation of an even degree, whose last term is negative, has at least two real roots, of different signs.*

For when zero is put for the unknown quantity x in the first member of the equation, the result will be negative. And (Art. 552,) if we give to x a value p sufficiently large, whether positive or negative, the result will be positive. For the first term, being an even power of x, will be positive, whether x is positive or negative; and if such a value p is assumed for x as to render this term greater than the sum of the others, all the terms taken together will furnish a positive result. The equation must therefore have a positive root between 0 and $+p$, and likewise a negative root between 0 and $-p$.

556. *An equation in which the signs of the terms are all plus, can have no positive root.*

For if a positive value be given to the unknown quantity, all the terms in the first member of the equation will be positive, and their sum can not equal zero.

557. *A complete equation, in which the signs of the terms are alternately plus and minus, can not have a negative root.*

For then it would have a positive root, if the signs of the alternate terms were changed. (Art. 550.) But this, as was shown in the preceding article, is impossible.

558. *If an equation has one imaginary root, of the form $a+b\sqrt{-1}$, it has another, of the form $a-b\sqrt{-1}$.*

For if $a+b\sqrt{-1}$ be substituted for x in the equation, and the powers developed, the terms in which $b\sqrt{-1}$ is raised to an even power will be free from the imaginary quantity $\sqrt{-1}$ (Art. 275,) while those in which the odd powers occur will contain it. We may then write the result in this form, $P+Q\sqrt{-1}=0$; where P stands for the sum of all the real terms, and $Q\sqrt{-1}$ for the sum of all the terms containing odd powers of b. And this equation can be satisfied only when $P=0$, and $Q=0$.

If we substitute $a-b\sqrt{-1}$ for x, the result will not differ from the preceding, except that each term containing an odd power of b will have its sign changed. The first member of the equation will therefore be $P-Q\sqrt{-1}$. And this is equal to zero, because $P=0$ and $Q=0$. Hence, $a-b\sqrt{-1}$ is a root of the equation.

Ex. 1. Find the cubic equation, of which two of the roots are 1 and $3+\sqrt{-2}$. Ans. $x^3-7x^2+17x-11=0$.

Ex. 2. One root of the equation

$$x^3-x^2+2=0,$$

is -1. What are the other roots? Ans. $1\pm\sqrt{-1}$.

Ex. 3. Two roots of the equation

$$x^4-2x^3+2x^2+2x-3=0,$$

are 1 and -1. What are the other roots?

Ans. $1\pm\sqrt{-2}$.

Ex. 4. Find the cubic equation of which two of the roots are -2 and $2-\sqrt{-1}$. Ans. $x^3-2x^2-3x+10=0$.

559. *If any polynomial of the form*

$$x^m\pm Ax^{m-1}\pm Bx^{m-2}\ldots\pm Tx\pm U$$

be multiplied by a binomial of the form $x-a$, there will be at least one more variation in the signs of the product than in those of the polynomial.

By a *variation* is meant the occurrence of any sign in a polynomial, *different* from the preceding sign. When two consecutive signs are *alike*, the combination is called a *permanence.*

If, for example,

$$x^7+2x^6+9x^5-x^4-5x^3+7x^2-4x-6,$$

in which there are three variations of sign, be multiplied by $x-3$, there will be at least four variations in the product.

The proposition may be most easily proved by attending first to a particular case.

If we take the preceding example, and complete the multiplication as below, it will be seen that there are *six* variations in the product.

$$\begin{array}{lllllllll}
x^7 & +2x^6 & +9x^5 & -x^4 & -5x^3 & +\ 7x^2 & -\ 4x & -\ 6 & \\
x & -3 & & & & & & & \\
\hline
x^8 & +2x^7 & +9x^6 & -\ x^5 & -5x^4 & +\ 7x^3 & -\ 4x^2 & -\ 6x & \\
 & -3x^7 & -6x^6 & -27x^5 & +3x^4 & +15x^3 & -21x^2 & +12x & +18 \\
\hline
x^8 & -\ x^7 & +3x^6 & -28x^5 & -2x^4 & +22x^3 & -25x^2 & +\ 6x & +18
\end{array}$$

But we may show, without regard to the particular values of the co-efficients, that there must be at least *four* variations in the signs of the product, because there are *three* in the signs of the multiplicand.

To do this, let us divide the given polynomial into groups of terms having the same sign, and pay regard in multiplying, only to the signs; thus

$$\begin{array}{ccc|cc|c|cc|c}
+ & + & + & - & - & + & - & - & \\
+ & - & & & & & & & \\
\hline
+ & + & + & - & - & + & - & - & \\
 & - & - & - & + & + & - & + & + \\
\hline
+ & * & * & - & * & + & - & * & +
\end{array}$$

The signs of the terms in the first partial product agree with the corresponding signs of the given polynomial; while the terms in the second partial product have the contrary signs: and they are set one place farther to the right; so that every sign in the first partial product differs from the one below it, except at the beginning of each group; where the upper and lower signs are always alike. Now, in obtaining the total from the partial products, if the signs of any two terms to be added are alike, the sign of the resulting

term must be the same. But if the two terms have different signs, the sign of the result may be plus or minus; and can not be determined without having reference to the co-efficients of the terms. It appears from what has just before been stated, that all the signs of the total product are thus ambiguous, except one at the beginning of each group. They are marked above by an asterisk.

Now as the sign at the beginning of each group in the multiplicand is the same as the sign below it in the total product, it is evident that while the signs vary from group to group in the multiplicand, there must likewise be at least one variation from the first sign of each group in the product to the first sign of the next group. And there is besides at least one variation between the last sign in the product and the first sign of the preceding group; so that the product must have at least one more variation than the multiplicand.

In the example before us, there is one variation in the multiplicand, from the first to the fourth term; and there is at least one, from the first to the fourth term in the product. Again, in the multiplicand, there is one variation from the fourth to the sixth term, and another from the sixth to the seventh. There are likewise in the product two corresponding variations; and there is one *additional* variation from the seventh to the ninth or last term. The same reasoning, evidently, may be applied to other examples.

In this demonstration, the polynomial has been supposed to be complete. To render the demonstration applicable to an incomplete polynomial, we have only to supply the place of every absent term with ± 0. The change in the form of the polynomial should be made, without altering the number of the variations. This may be done, by giving to zero, in every case, the sign of the term which precedes it. For example, the incomplete polynomial x^5-2x^2+1, should be written in the form $x^5+0+0-2x^2-0+1$; where the original number of variations is preserved.

We shall have occasion to employ the theorem just proved, in the demonstration of the following proposition; which is commonly called *Descartes' Rule.*

560. *The number of positive roots, in any equation, can not be greater than the number of variations in the signs of its terms.* And

The number of negative roots, in any COMPLETE *equation, can not exceed the number of permanences in its signs.*

Every equation having positive roots, as a, b, c, &c. may be written thus

$$(x^n+Hx^{n-1}+Kx^{n-2} \text{ \&c.})(x-a)(x-b)(x-c) \text{ \&c.} =0;$$

the first member being resolved into factors, as in Art. 538.

For example,

$$x^5-6x^4+12x^3-4x^2-13x+10=0,$$

which will be found, by trial, to have the positive roots 1 and 2, may be written in the form,

$$(x^3-3x^2+x+5)(x-1)(x-2)=0,$$

the polynomial factor being the product of all the binomial factors of which the equation consists (Art. 538,) except $x-1$ and $x-2$, which belong to the positive roots. Now whether there be any variation in the signs of the terms of the polynomial $x^n+Hx^{n-1}+Kx^{n-2}$ &c. or not, there will be *one* at least after multiplying it by $x-a$, *two* after multiplying further by $x-b$, *three* after multiplying by $x-c$, &c. (Art. 559.) By these multiplications, the first member of the above equation is reduced to the form it must have had, before being resolved into factors. And we see that there must be at least as many variations in the signs of its terms, as there are positive roots, a, b, c, &c.

The first part of the proposition is therefore true. The second remains to be proved.

If $$x^m+Ax^{m-1}+Bx^{m-2}+Cx^{m-3} \text{ \&c.} =0$$

be a *complete* equation, the number of its negative roots is equal to the number of positive roots belonging to the equation

$$x^m-Ax^{m-1}+Bx^{m-2}-Cx^{m-3} \text{ \&c.} =0 \quad \text{(Art. 550,)}$$

And this number can not exceed the number of variations in the signs of the terms; as appears from the preceding demonstration. But the number of variations in the second equation, equals the number of permanences in the first. For it is evident that when two consecutive terms have unlike signs in one of the equations, the corresponding terms have like signs in the other. Hence we conclude that the number of negative roots, in the first equation, can not exceed the number of permanences in its signs.

It is evident that this reasoning will also apply to any incomplete equation, whose terms contain alternately odd and even powers of x; but to no other. Thus it will apply to the equation

$$x^8+6x^5-7x^4-3x+12=0;$$

but not to the equation

$$x^6-x^5+2x^3-5x^2+7x+3=0;$$

in which there are two odd powers, x^5 and x^3, occurring in immediate succession.

It is easy to show, by an example, that an *incomplete* equation may have more negative roots than permanences in its signs.

Thus the equation $\quad x^3-7x-6=0$

has but one permanence, while it has two negative roots, -1 and -2.

But the second part of Descartes' Rule may be applied to all incomplete as well as complete equations, if the place of each deficient term be first supplied with ± 0, so as to render the equations complete in form.

Take for example the preceding equation, namely

$$x^3-7x-6=0;$$

which, in its present form, has only one permanence. Supplying the place of the absent term, we have

$$x^3\pm 0-7x-6=0,$$

and whether $+0$ or -0 be used, the equation now has two permanences; that is, it has as many permanences as negative roots.

Ex. 1. The equation

$$x^3-2x^2-x+2=0$$

has three real roots. How many of them are positive?

Ans. *Two.*

The equation can not have more than one negative root, as it has only one permanence. And since the whole number of real roots is three, there can not be less than two positive roots. Neither can there be more than two such roots; for the equation has only two variations.

Ex. 2. The equation

$$x^3+x^2+100=0$$

has one negative root, as appears by Art. 554. Is either of its remaining roots negative?

If we supply the place of the deficient term with $+0$, the equation becomes

$$x^3+x^2+0+100=0.$$

Here there are three permanences; from which it would appear that there may be three negative roots. But we are at liberty to supply the place of the absent term with -0. The equation will then be

$$x^3+x^2-0+100=0;$$

in which there is only one permanence. And hence it appears that there can not be more than one negative root.

By this example we see, that in supplying the place of an absent term with ± 0, an advantage is gained by using the sign which will produce the smallest number of permanences.

Ex. 3. The equation

$$x^4-5x^2+4=0$$

has four real roots. How many of them are positive?

Ex. 4. The equation

$$x^5-x^4-5x^3-x+6=0,$$

has three real roots. How many of these are positive?

561. It has been remarked in Art. 539, that the roots of an equation are not always unequal. In order to determine whether a proposed equation has any equal roots, we have occasion to employ a property of derived polynomials, which we will here demonstrate.

Let the polynomial

$$x^m+Ax^{m-1}+Bx^{m-2}\ldots\ldots+Tx+U$$

be denoted by X; and let its first derived polynomial, which is

$$mx^{m-1}+(m-1)Ax^{m-2}+(m-2)Bx^{m-3}\ldots+T,$$

be denoted by X'.

This, according to Art. 545, is the co-efficient of the first power of y in the result obtained by substituting $x+y$ for x in the polynomial X.

Now (Art. 538,) X may be expressed in the form

$$(x-a)(x-b)(x-c)\ldots\ldots(x-l),$$

where $a, b, c, \ldots l$, are the values of x which will satisfy the equation $X=0$; that is, are the m roots of this assumed equation. And if in this new form of X, we substitute for x, $x+y$ or what is the same thing $y+x$, the result is

$$(y+x-a)(y+x-b)(y+x-c)\ldots\ldots(y+x-l).$$

The factors here may be regarded as binomials, of which the first term is y, and the second terms are $x-a$, $x-b$,

$x-c, \ldots x-l$. Then (Art. 491,) if the multiplication be performed, the co-efficient of the first power of y in the result will be the sum of the products of the m quantities $x-a$, $x-b$, $x-c, \ldots x-l$, taken $m-1$ and $m-1$. And it is evident that in order to obtain these products, we have only to divide X, in each case, by the omitted factor; first by $x-a$, next by $x-b$, and so on.

Hence we infer that

$$X' = \frac{X}{x-a} + \frac{X}{x-b} + \frac{X}{x-c} \cdots + \frac{X}{x-l}, \text{ That is,}$$

A polynomial of the (m)th degree being given, its first derived polynomial is equal to the sum of the products of its simple factors, taken $m-1$ and $m-1$: or is equal to the sum of the quotients obtained by dividing the given polynomial by each of its simple factors.

For example, let the given polynomial be

$$x^4 - x^3 - 7x^2 + x + 6.$$

If this be made equal to zero, the roots of the equation thus formed, will be found, by trial, to be $1, -1, -2, 3$. The simple factors, of which the polynomial consists, must therefore be $x-1$, $x+1$, $x+2$, $x-3$, (Art. 538.) Now the first derived polynomial of the given expression is

$$4x^3 - 3x^2 - 14x + 1.$$

And we shall also find this to be the sum of the products,

$$(x+1)(x+2)(x-3),\ (x-1)(x+2)(x-3),\ (x-1)(x+1)(x-3),$$
$$(x-1)(x+1)(x+2).$$

or the sum of the quotients,

$$\frac{X}{x-1},\ \frac{X}{x+1},\ \frac{X}{x+2},\ \frac{X}{x-3};$$

where $\qquad X = x^4 - x^3 - 7x^2 + x + 6.$

We may now, by the aid of the preceding theorem, explain the method by which the equal roots of an equation are determined.

562. Let the equation be

$$x^m + Ax^{m-1} + Bx^{m-2} \ldots + Tx + U = 0;$$

and let it have n roots equal to a, p roots equal to b, q roots equal to c, &c. Then (Art. 538,) if the first member be denoted by X,

$$X = (x-a)^n (x-b)^p (x-c)^q \text{ \&c.}$$

And by the preceding theorem,

$$X'=\frac{X}{x-a}+\frac{X}{x-a}\ \&c.+\frac{X}{x-b}+\frac{X}{x-b}\ \&c.+\frac{X}{x-c}+\frac{X}{x-c}\ \&c.+\&c.$$

the number of equal quotients being n in the first case, p in the second, q in the third, &c. This equation, when simplified, becomes

$$X'=n\frac{X}{x-a}+p\frac{X}{x-b}+q\frac{X}{x-c}\ \&c.$$

Or

$$\begin{aligned}X'=&\,n(x-a)^{n-1}(x-b)^p(x-c)^q\ \&c.\\ &+p(x-a)^n(x-b)^{p-1}(x-c)^q\ \&c.\\ &+q(x-a)^n(x-b)^p(x-c)^{q-1}\ \&c.\\ &+\ \&c.\end{aligned}$$

If we compare the preceding values of X and X', we shall see that they are both divisible by

$$(x-a)^{n-1}(x-b)^{p-1}(x-c)^{q-1}\ \&c.$$

And this is their greatest common divisor. For if not, then the quotient

$$n(x-b)(x-c)\ \&c.+p(x-a)(x-c)\ \&c.+q(x-a)(x-b)\ \&c.+\&c.$$

which is obtained by taking out this divisor from X', must be divisible by one of the factors of X. Now each of the simple factors, $x-a$, $x-b$, $x-c$, &c. contained in X, is a divisor of all the terms in that quotient, except one. The whole quotient therefore is divisible by neither of these factors.

Hence,

If an equation has equal roots, the first member and its derived polynomial have a common divisor: And their greatest common divisor is the product of all the factors corresponding to the equal roots, each raised to a power one degree lower than in the given equation.

It is also evident that

If an equation has no equal roots, the first member and its derived polynomial have no common divisor.

For in that case, n, p, q, &c. are each equal to unity, and the expression for the greatest common divisor is reduced to $(x-a)^0(x-b)^0(x-c)^0$ &c., which is simply 1. (Art. 224.)

Ex. 1. What are the equal roots of the equation

$$x^5+3x^4-x^3-7x^2+4=0\,?$$

The derived polynomial of the first member is

$$5x^4+12x^3-3x^2-14x;$$

and it will be found that x^2+x-2 is the greatest common measure of the two expressions. By solving the equation $x^2+x-2=0$, we obtain the roots, 1 and -2. Hence (Art. 538,) x^2+x-2 may be expressed in the form $(x-1)(x+2)$. We infer then, by the preceding theorem, that the original equation contains the factors $x-1$, $x+2$, each raised to the second power; that is, the equation has two roots equal to 1, and two others equal to -2.

If we divide the first member of the equation by $(x-1)^2(x+2)^2$, we shall obtain the remaining factor, namely $x+1$. Hence -1 is the fifth root of the equation.

Ex. 2. Find the equal and the unequal roots of the equation

$$x^5+4x^4-14x^2-17x-6=0.$$

Ans. $-1, -1, -1;\ 2, -3$.

Ex. 3. Find the roots of the equation

$$x^5+x^4-14x^3+26x^2-19x+5=0.$$

Ans. $1, 1, 1, 1;\ -5$.

Ex. 4. Find the roots of the equation

$$x^6+2x^5-12x^4-14x^3+47x^2+12x-36=0.$$

Ans. $2, 2;\ -3, -3;\ 1, -1$.

563. If the original equation, in the preceding article, be divided by the common measure $(x-a)^{n-1}(x-b)^{p-1}(x-c)^{q-1}$ &c. the result is $(x-a)(x-b)(x-c)$ &c. $=0$; which contains all the different roots of the original equation, but no equal roots.

Sturm's Theorem.

564. This important theorem receives its name from an eminent French mathematician by whom it was discovered in 1829. It may be stated as follows.

Let X represent the first member of an equation of any degree; and suppose the equation to have *no equal roots.* Also let X' be the first derived polynomial of X, (Art. 545.) Apply to X and X' the rule in Art. 476, for finding the greatest common measure; but with this alteration, that the signs of each remainder are to be changed, before it is used as a divisor. In other words, divide X by X', and disregarding the quotient, let the remainder, *after all its signs are changed,*

be denoted by X''. Divide X' by X'', and let X''' represent the remainder with its signs changed. In like manner divide X'' by X'''; and so on. As each successive remainder is of a lower degree, with respect to x, than the preceding, there will at last be found a remainder free from the unknown quantity x; that is, a numerical remainder. And this remainder can not be zero; for then would X and X' have a common divisor (Art. 476,) and the equation $X=0$, would have equal roots (Art. 562); which is contrary to the supposition. Let the last remainder, taken with the contrary sign, be denoted by X^r. Then if we substitute any number p for x, in each of the expressions X, X', X'', X''',...X^r, and mark the sign of the resulting quantity; and in like manner substitute any other number q for x, and mark the sign of each of the results; the two series of signs thus obtained, will be such that,

The difference between the number of variations in the first series and that in the second, is equal to the number of real roots of the given equation, comprized between p and q.

565. The demonstration of this theorem depends on the four following propositions.

Prop. I. *No number, substituted for x in the series of quantities X, X', X''...X^r, can reduce any two consecutive ones to zero.*

If we take Q', Q'', &c. to represent the quotients obtained by dividing X by X', X' by X'' &c. and remember that the remainders of these divisions are $-X''$, $-X'''$, &c. we shall have the following equations,

$$X = Q'X' - X''$$
$$X' = Q''X'' - X'''$$
$$X'' = Q'''X''' - X''''$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$X^{r-2} = Q^{r-1}X^{r-1} - X^r$$

Now if two successive quantities, as X' and X'', become zero, for any value of x, the next quantity X''' must also become zero; as appears from the second equation above. And since the values of X'' and X''' are zero, that of X'''' must likewise be zero; as we see by the third equation. By proceeding in this way, from one equation to another, we at length infer that X^r equals zero; which has been shown, in the preceding article, to be impossible.

Prop. II. *When one of the quantities between X and X^r, in the series X, X', X'',...X^r, becomes zero, for a particular value of x, the preceding and the following quantity must have opposite signs.*

For if one of these intermediate quantities, as X''', equals zero, one of the above equations, namely $X''=Q''X'''-X''''$, will become $X''=-X''''$; showing that X'' and X'''' have equal values, but with *opposite signs.*

Prop. III. *If one of the quantities between X and X^r becomes zero for a particular value of x, then for the values of x a little greater and a little less than this, the sign of that quantity will form a permanence with the sign of one of the two adjacent quantities, and a variation with the sign of the other.*

Suppose that $X''=0$, when $x=a$: it is required to prove that for any value of x a little greater or a little less than a, the three consecutive quantities X', X'', X''', will have values whose signs present one variation and one permanence.

Since a is a root of the equation $X''=0$, it can not be a root of either of the equations $X'=0$, and $X'''=0$, (Prop. I.) Of all the roots of these two equations, let that which is nearest in value to a, be denoted by b; and let h be any quantity less than the difference between a and b. Then there is no value of x between $a+h$ and $a-h$, which will satisfy either of the two preceding equations. Now suppose x to vary in value by imperceptible degrees from $a-h$ to $a+h$: since X' will not be reduced to zero, during this variation, all its values must be of the same sign; for a quantity varying by insensible degrees, can not change its sign without becoming zero. All the values of X''' must likewise be of one sign, while x varies between the limits $a-h$ and $a+h$. Now when $x=a$, X' and X''' have opposite signs, (Prop. II.) Hence, while x varies from $a-h$ to $a+h$, X' has constantly one sign, and X''' has the opposite sign. The sign of X'' therefore, whatever it may be, must form a permanence with the sign of one of the quantities X' and X''', and a variation with the sign of the other.

Prop. IV. *If (a) is a root of the equation $X=0$, and any number a little less than (a) be substituted for (x) in X and X', the results will have opposite signs; but if a number a little greater than (a) be substituted, the results will have the same sign.*

Let Z and Z' represent the results obtained by substituting $a+h$ for x in X and X'. Then (Art. 545,)

$$Z=A+A'h+\frac{A''}{2}h^2+\frac{A'''}{2.3}h^3+ \&c.$$

and $$Z'=A'+A''h+\frac{A'''}{2}h^2+ \&c.$$

where A and A' are the results from substituting a for x in X and X', (Art. 545); and A' is the derived polynomial of A. But $A=0$, since a is a root of the equation $X=0$. Therefore

$$Z=A'h+\frac{A''}{2}h^2+\frac{A'''}{2.3}h^3+ \&c.$$

$$=h(A'+\frac{A''}{2}h+\frac{A'''}{2.3}h^2+ \&c.)$$

Now A' is different from zero, (Prop. I;) and h may be supposed so small as to render all the terms after A' nearly equal to zero. Their sum may thus be made less than A'; so that the sign of A' shall be the sign of the aggregate of all the terms. Then Z will have the same sign as hA', or as A'. We may also suppose h so small that the value of Z' shall have the same sign as the first term A'. *The signs of Z and Z' will therefore be alike.*

But if $a-h$ be substituted for x, and the results be denoted as before, by Z and Z'; then

$$Z=-A'h+\frac{A''}{2}h^2-\frac{A'''}{2.3}h^3+ \&c.$$

$$=h(-A'+\frac{A''}{2}h-\frac{A'''}{2.3}h^2+ \&c.)$$

and $$Z'=A-A''h+\frac{A'''}{2}h^2- \&c.$$

from which it appears that for small values of h, Z has the same sign as $h(-A')$, or as $-A'$; while Z' has the same sign as A'. In this case therefore *Z and Z' have opposite signs.*

Thus it is proved that if h be a very small quantity, and $a+h$ be substituted for x in X and X', the results will have the same sign; but if $a-h$ be substituted, the results will have contrary signs.

566. *Sturm's Theorem Demonstrated.*—Let p and q be two numbers between which are comprized all the roots of the equations

$$X=0,\ X'=0,\ X''=0, \ldots X^{r-1}=0;$$

and let q be algebraically less than p; that is, nearer to $-\infty$.

If q be substituted for x, in the expressions X, X', X'', &c. the signs of the resulting values will present a certain number of permanences and variations. Now suppose the value of x in these expressions to begin at q and increase by insensible degrees. The values of the expressions will thus be changed; but they will have the same signs as at first, till x attains a value which reduces one of them to zero. This expression will then change its sign; and the original series of signs will thus be altered. The vanishing expression can not be X^r, since this is a *number*, independent of x. We will first suppose it to be X. Then as the signs of X and X' were unlike before x reached the value which reduces X to zero, and are alike after x has passed this value (Prop. IV,) a variation has been lost, at the beginning of the series of signs, by being changed into a permanence.

Next suppose either of the intermediate expressions X', X'',...X^{r-1}, to vanish for any particular value of x, and to change its sign as x passes this value. *This* change will not affect the number of variations in the series of signs: for (Prop. III,) it must take place between two opposite signs; and the three consecutive signs must form one permanence and one variation, both before and after the change. Thus it appears that as x increases in value, one variation will be lost, by being changed into a permanence, whenever x passes a root of the equation $X=0$; and in no other case will the number of variations be altered: so that when x arrives at the value p, the number of variations lost, will be just equal to the whole number of real roots of the equation $X=0$.

And from this demonstration it is evident that if p' and q' be *any* assumed values of x, the difference between the number of variations in the signs of the quantities X, X', X'',..X^r, for the value p', and the number for q', is equal to the number of real roots of the equation $X=0$, that are comprized between p' and q'.

When the *whole* number of real roots of an equation is to be determined, we may take for p and q the values $+\infty$ and $-\infty$; since every real root must be included between these limits.

The number of *positive* roots may be found by supposing $p=\infty$ and $q=0$; and the number of negative roots, by supposing $p=0$ and $q=-\infty$.

When an equation has *equal* roots, we may find, by the method of Art. 563, another equation containing all the different roots of the given one, but without the repetition of any; and we may then apply the theorem of Sturm to this new equation.

Before proceeding to illustrate the theorem by examples, it is important to observe that in deducing X'', X''', &c. from X and X', as in finding the greatest common measure of two quantities, we may apply the principle of Art. 477; but the number by which we divide or multiply, must always be *positive*, so that the signs of the quantities X'', X''', &c. may not be affected.

Ex. 1. Find the number of real roots of the equation

$$x^3-2x^2-6x+4=0.$$

Representing the first member by X, and its derived polynomial by X', we have

$$X'=3x^2-4x-6, \text{ (Art. 545.)}$$

We now proceed as in finding the greatest common measure of X and X'; that is, we multiply X by 3 and divide by X', as follows.

$$\begin{array}{r|l}
3x^3-6x^2-18x+12 & 3x^2-4x-6 \\
\underline{3x^3-4x^2-\ \ 6x\qquad} & \overline{x\ -1} \\
(-2x^2-12x+12) & \\
-3x^2-18x+18 & \\
\underline{-3x^2+\ 4x+\ 6} & \\
(-22x+12) & \\
-11x+\ 6 &
\end{array}$$

Here the remainder $-2x^2-12x+12$ is changed to $3x^2-18x+18$, by dividing it by $+2$, and then multiplying by $+3$, (Art. 477.) This change is made, to enable us to carry out the division. When the division is completed, we have the remainder $-22x+12$. To simplify this, we divide it by $+2$. (Art. 477.) The result, with its sign changed is $11x-6$; which we denote by X''. Proceeding in a similar way with X' and X'', we find the remainder -441; whence $X'''=+441$.

We have then this series of equations

$$X\ =x^3-2x^2-6x+4$$
$$X'\ =3x^2-4x-6$$
$$X''=11x-6$$
$$X'''=+441$$

When $+\infty$ or $-\infty$ is substituted for x, the signs of X, X', &c. must depend on the signs of their first terms, (Art. 552.)

Hence, if $x=-\infty$, the signs of X, X', X'', X''', will be $-+-+$, forming 3 variations.

And, if $x=+\infty$, the signs . . . . will be $++++$, forming *no* variation.

The given equation therefore has 3 real roots.

To determine whether they are positive or negative,

Suppose $x=0$: the signs of X, X', X'', X''', will then be $+--+$, forming 2 variations.

Hence, one of the roots is negative, and two are positive.

By assuming different values for x, we may determine very nearly the situation of the roots. Thus,

If $x=1$,	the four signs will be	$--++$,	forming	1 variation.
$x=2$,	"	$--++$	"	1 "
$x=3$,	"	$-+++$	"	1 "
$x=4$,	"	$++++$	"	0 "

And for all higher values of x, to $+\infty$, the four signs will continue to be positive.

It appears that one variation is lost between $x=0$ and $x=1$; and another between $x=3$ and $x=4$. Hence one of the positive roots lies between 0 and 1; and the other between 3 and 4. Again,

If $x=-1$,	the four signs will be	$++-+$,	forming	2 variations
$x=-2$,	"	$\pm+-+$,	"	2 or 3 "
$x=-3$,	"	$-+-+$,	"	3 "

And for all higher negative values, up to $-\infty$, the four signs will continue to be the same as in the last case.

Here it is seen that when $x=-2$, $X=0$; from which it follows that -2 is the negative root of the proposed equation. We have given to X, in this case, the ambiguous sign $\pm$; leaving it undetermined, as it must be, whether the number of variations is 2 or 3. There is one variation gained between $x=-1$ and $x=-3$; but we can not say that it is gained

between $x=-1$ and $x=-2$, or *between* $x=-2$ and $x=-3$: it is gained when x, in varying, *passes through* the value -2.

Ex. 2. How many real roots has the equation

$$2x^3+2x^2-8x-7=0?$$

Here we have

$$X\ =2x^3+2x^2-8x-7$$
$$X'\ =3x^2+2x-4=\ (6x^2+4x-8)\div 2,\ \text{(Art. 477.)}$$
$$X''\ =52x+55$$
$$X'''=+7461$$

When $x=-\infty$,	our series of signs is	$-+-+$,	giving	3	variations
" $x=+\infty$,	"	$++++$,	"	0	"

Hence the equation has 3 real roots.

By assuming different values for x, we find that

When $x=-3$,	the series of signs is	$-+-+$,	giving	3	variations
" $x=-2$,	"	$++-+$	"	2	"
" $x=-1$,	"	$+-++$	"	2	"
" $x=0$,	"	$--++$	"	1	"
" $x=1$,	"	$-+++$	"	1	"
" $x=2$,	"	$++++$	"	0	"

The equation therefore has one negative root between -2 and -3, another between 0 and -1, and a positive root between 1 and 2.

Ex. 3. How many real roots has the equation

$$x^4+4x^3+x^2-16x-18=0?$$

In this case,

$$X\ =x^4+4x^3+x^2-16x-18$$
$$X'\ =2x^3+6x^2+x-8\quad \text{(Art. 477.)}$$
$$X''\ =5x^2+25x+28$$
$$X'''=-49x-72$$
$$X''''=-4948$$

When $x=-\infty$,	the series of signs is	$+-++-$,	giving	3	variations.
" $x=-2$	"	$+--+-$	"	3	"
" $x=-1$	"	$--+--$	"	2	"
" $x=0$	"	$--+--$	"	2	"
" $x=1$	"	$-++--$	"	2	"
" $x=2$	"	$+++--$	"	1	"
" $x=\infty$	"	$+++--$	"	1	"

The equation therefore has only two real roots; a negative one between -1 and -2, and a positive one between 1 and 2. It has of course two other roots that are imaginary.

Ex. 4. How many real roots has the equation

$$x^3+3x^2+7x+4=0?$$

Ans. One between 0 and -1.

Ex. 5. How many real roots has the equation

$$x^4-4x^2-8x-4=0? \qquad \text{Ans. Two.}$$

Ex. 6. How many real roots has the equation

$$2x^3-5x-8=0?$$

Ans. One between 2 and 3.

Ex. 7. How many real roots has the equation

$$x^4-7x^2-2x+2=0? \qquad \text{Ans. Four.}$$

Elimination.

567. The methods of elimination explained in Sect. VIII, are particularly adapted to simple equations. They can not often be conveniently employed, when the proposed equations contain powers and products of the unknown quantities. A method will now be given, which is applicable to equations of every degree.

Let the proposed equations be

$$M=0, \ N=0;$$

where M and N stand for any polynomials containing x and y. If we apply to M and N the process for finding their greatest common measure, (Art. 476,) and denote the successive quotients by Q, Q', &c. and the remainders by R, R', &c. then since

$$M=NQ+R, \ N=RQ'+R', \ R=R'Q''+R'', \text{ \&c.}$$

and since $M=0$, and $N=0$, we see that R and each of the following remainders must be equal to zero. Now if the polynomials M and N have been arranged according to the powers of x, each remainder must be of a lower degree, with respect to this quantity, than the preceding one; and there must finally be a remainder which is entirely free from x. If we make this remainder equal to zero, we shall have an equation which contains only the unknown quantity y, and which has been obtained by eliminating x from the two given equations.

The value of x may be obtained, in terms of y, by making the last remainder but one equal to zero.

Ex. 1. Eliminate x from the equations

$$4x^3-y^2x-y^3-1=0,$$
$$2x^2+yx-a=0.$$

Applying the method just explained, we proceed as follows.

$$\begin{array}{l|l}
4x^3-y^2x-y^3-1 & 2x^2+yx-a \\
\underline{4x^3+2yx^2-2ax} & 2x-y \\
\quad -2yx^2+(2a-y^2)x-y^3-1 & \\
\quad \underline{-2yx^2-y^2x\ \ +ay} & \\
\qquad\qquad 2ax-y^3-ay-1= \text{1st remainder.} &
\end{array}$$

$$\begin{array}{l|l}
2ax^2+ayx-a^2 & 2ax-y^3-ay-1 \\
\underline{2ax^2-(y^2+ay+1)x} & x+y^2+2ay+1 \\
\quad (y^2+2ay+1)x-a^2 & \\
2a(y^2+2ay+1)x-2a^3 & \\
\underline{2a(y^2+2ay+1)x-(y^2+ay+1)(y^2+2ay+1)} & \\
\qquad (y^2+ay+1)(y^2+2ay+1)-2a^3= \text{2d rem.} &
\end{array}$$

Ans. $y^4+3ay^3+2(a^2+1)y^2+3ay-2a^3+1=0.$

The value of x, in terms of y, is $\dfrac{y^2+ay+1}{2a}$.

For the sake of avoiding fractions, the partial remainder $(y^2+2ay+1)x-a^2$ is here multiplied by $2a$, (Art. 477,) before completing the division. The first divisor also, before being divided by the first remainder, is multiplied by a.

Ex. 2. Eliminate x from the equations

$$x^2+y-1=0$$
$$2xy-y-1=0$$

Ans. $4y^3-3y^2+2y+1=0.$

Ex. 3. Eliminate x from the equations

$$2x^3-xy^2+y^3-1=0$$
$$x^2-xy\ \ +y^2-1=0$$

Ans. $y^6-4y^5+3y^4+5y^3-6y^2+1=0.$

Ex. 4. Eliminate x from the equations

$$x^3+y^3-a=0$$
$$x^2+xy+y^2-b=0$$

Ans. $4y^6-6by^4-4ay^3+3b^2y^2+3aby+a^2-b^3=0.$

Ex. 5. Eliminate x from the equations

$$x^2+ay-b=0$$
$$y^2+cx-d=0$$

Ans. $y^4-2dy^2+ac^2y+d^2-bc^2=0.$

When we have three equations containing three unknown quantities, we may first eliminate one of the quantities, by combining either of the equations with the other two; and having thus obtained two new equations containing only two unknown quantities, we may proceed with these as in the former case.

A similar method may be pursued in cases where there are four or more equations given.

Ex. 6. Eliminate x and y from the equations

$$xy-z-10=0$$
$$xz-y+11=0$$
$$yz-x+14=0$$

Ans. $z^5+10z^4-2z^3-174z^2-316z-144=0.$

Ex. 7. Eliminate x and y from the equations

$$x+xy-a=0$$
$$y+yz-b=0$$
$$z+zx-c=0$$

Here we may eliminate x by combining the first and third equations; and we must then combine the new equation with the second, in order to eliminate y.

Ans. $(a+1)z^2+(a+b-c+1)z-c(b+1)=0.$

Ex. 8. Eliminate x and y from the equations

$$y(x+y-z)=a$$
$$z(y+z-x)=b$$
$$x(z+x-y)=c$$

Ans. $8z^8-4(a+5b+2c)z^6+2(2ab+9b^2+5bc+c^2)z^4$
$-b^2(a+7b+3c)z^2+b^4=0.$

SECTION XX.

RESOLUTION OF EQUATIONS.

ART. **568.** THE real roots of equations are either *rational* or *irrational*. *Rational* or *commensurable* roots are such as can be *exactly* expressed in numbers. *Irrational* or *incommensurable* roots are those which can only be expressed *approximately*.

We will first explain the method of finding

RATIONAL ROOTS.

569. *An equation whose co-efficients are whole numbers, and that of the first term unity, can have only whole numbers for its rational roots.*

For let the equation be

$$x^m+Ax^{m-1}+Bx^{m-2}\ldots+Tx+U=0;$$

where $A, B, \ldots T, U$, are whole numbers; and if possible let the irreducible *fraction* $\frac{a}{b}$ be a root of the equation.

Then (Art. 537,) $\frac{a^m}{b^m}+A\frac{a^{m-1}}{b^{m-1}}+B\frac{a^{m-2}}{b^{m-2}}\cdots+T\frac{a}{b}+U=0.$

If we multiply this by b^{m-1} and transpose, we obtain

$$\frac{a^m}{b}=-Aa^{m-1}-Ba^{m-2}b\ldots-Tab^{m-2}-Ub^{m-1}.$$

Here the first member $\frac{a^m}{b}$ is an irreducible fraction; for since $\frac{a}{b}$ is in its lowest terms, a and b have no common divisor; and therefore a^m and b can have no common divisor. But the second member is a whole number, since all its terms are integral. We have then an irreducible fraction equal to a whole number; which is absurd.

570. We proceed therefore to show how the *integral* roots of an equation may be determined, supposing the co-

efficients as before to be whole numbers, and that of the first term unity.

Let the given equation be

$$x^4+Ax^3+Bx^2+Cx+D=0.$$

From Art. 538, it appears that if any number a be a root of this equation, the first member can be resolved into two factors, one of which is $x-a$, and the other is a polynomial of the third degree. Let this be represented by

$$x^3+A'x^2+B'x+C'.$$

Then $x^4+Ax^3+Bx^2+Cx+D=(x-a)(x^3+A'x^2+B'x+C')$

$$=x^4+(A'-a)x^3+(B'-aA')x^2+(C'-aB')x-aC'$$

As the two members of this equation must be precisely alike, we see that

(1) $A'-a\ =A$

(2) $B'-aA'=B$

(3) $C'-aB'=C$

(4) $-aC'=D$

To obtain now the integral roots of the given equation, we have only to find every integral value of a, which will satisfy these four conditions.

Since A, B, C and D are supposed to be integers, it is evident from the equations (1), (2), (3), that if a be an integer, A', B', C', are also integers. Now the four equations of condition, taken in the contrary order, may be changed into the following.

$$-C'=\frac{D}{a}$$

$$-B'=\frac{C-C'}{a}$$

$$-A'=\frac{B-B'}{a}$$

$$-1=\frac{A-A'}{a}$$

Whence it appears that

The quotient of D divided by a must be a whole number.

And if this quotient be added to C, and the sum divided by a, the new quotient must be a whole number.

And again, if this be added to B, and the sum divided by a, the quotient must be a whole number.

And finally, if this be added to A, and the sum divided by a, the quotient must be -1.

It follows from the first of these conditions, that all the integral roots of the given equation are comprized among the divisors of its last term.

If then we substitute each of these divisors for a, and ascertain by trial which of them will satisfy the above conditions, the required roots will be determined.

But after obtaining, in this manner, *one* integral root, as a, we may depress the given equation to a lower degree, by dividing by $x-a$. The result will be $x^3+A'x^2+B'x+C'=0$. Now the values of $-C'$, $-B'$, $-A'$, were found in testing the root a; so that no actual division is required, to obtain this new equation. The roots of the cubic equation are the same as those of the original equation, with the exception of a. (Arts. 535, 538.) Then instead of seeking a second integral root of the given equation, we may proceed to find an integral root of the new equation; which is a simpler problem. The same method may be pursued in finding all the other integral roots.

The method which has now been explained with reference to an equation of the fourth degree, is obviously applicable to equations of other degrees. Hence we have the following general

RULE.

To find the integral roots of any equation, in which the co-efficients are integers, and that of the first term unity,

Divide the last term of the equation by one of its factors, which may be denoted by (a); add the result to the co-efficient of x, the unknown quantity, and divide the sum by (a). The quotient should be a whole number. If so, add it to the co-efficient of x^2, and divide the sum by (a). If the quotient is a whole number, add it to the co-efficient of x^3 and divide by (a). And so proceed till all the co-efficients of the equation have been employed, except that of the first term. The last quotient should be -1. If this is the case, the number (a) must be a root of the equation. But if (a) fails to render any of the quotients integral, or to make the last quotient -1, it can not be a root, and a new number must be tried.

Having obtained one integral root, to find another, change the signs of the quotients obtained in trying the root already found; then make these quotients the co-efficients of an equation of the next lower degree; and proceed with this as with the original equation. And in the same manner find all the remaining integral roots.

If any term is wanting in the given equation, zero must be used in the place of its co-efficient.

It is generally best to try the smaller divisors first, beginning always with 1 and -1.

When $a=1$, the above equations of condition may be reduced to these.

$$\begin{aligned} A'&=A+1 \\ B'&=B+A' \\ C'&=C+B' \\ D+C'&=0, \text{ or } D=-C'. \end{aligned}$$

And when $a=-1$,

$$\begin{aligned} A'&=A-1 \\ B'&=B-A' \\ C'&=C-B' \\ D-C'&=0, \text{ or } D=C'. \end{aligned}$$

These formulas, with the corresponding ones for equations of other degrees than the fourth, are more convenient than the preceding rule, for trying the *particular* divisors 1 and -1.

Ex. 1. Find the integral roots of the equation

$$x^5-6x^4-2x^3+36x^2+x-30=0.$$

Trying the divisor 1 by the method just indicated, we find

$$\begin{aligned} A'&=-\ 6+\ 1=-5 \\ B'&=-\ 2-\ 5=-7 \\ C'&=\ 36-\ 7=\ 29 \\ D'&=\ \ 1+29=\ 30 \\ -30+D'&=-30+30=\ 0. \end{aligned}$$

Here we have *five* equations of condition, because the given equation is of the fifth degree; and as they are all satisfied, 1 is a root of this equation.

If the given equation be divided by $x-1$, the result will be

$$x^4-5x^3-7x^2+29x+30=0;$$

in which the co-efficients are the values of A', B', C', D', as found above.

Trying the divisor 1 in this equation, we find $A'=-4$, $B'=-11$, $C'=18$, $D+C'=30+18=48$. And as the sum of D and C' is not zero, 1 can not be a root.

Trying -1 we find

$$\begin{aligned} A' &= -5-1=-6 \\ B' &= -7+6=-1 \\ C' &= 29+1=30 \\ D-C' &= 30-30=0 \end{aligned}$$

Therefore -1 is a root; and the next lower equation is

$$x^3-6x^2-x+30=0.$$

Here if the divisor -1 be tried, it will be found unsatisfactory.

Trying 2, which is a divisor of 30, by the preceding rule, we obtain

$$\frac{30}{2}=15$$

$$\frac{-1+15}{2}=7$$

$$\frac{-6+7}{2}=\frac{1}{2}.$$

As the last quotient is not -1, 2 is not a root.

If we try -2, we shall find the successive quotients to be -15, 8, -1. Therefore -2 is a root; and the next lower equation is $\qquad x^2-8x+15=0.$

The two remaining roots are 3 and 5; as may be found by resolving this quadratic in the usual way.

Ex. 2. Find the roots of the equation

$$x^4-81x^2-310x-150=0.$$

Here the divisors of the last term are $1, -1, 2, -2, 3, -3, 5, -5$, &c. It will be seen by trial that none of the first seven of these is a root. But if we try -5, we obtain

$$\frac{-150}{-5}=30$$

$$\frac{-310+30}{-5}=56$$

$$\frac{-81+56}{-5}=5$$

$$\frac{0+5}{-5}=-1.$$

Therefore -5 is a root of the given equation; and the next lower equation is

$$x^3-5x^2-56x-30=0.$$

If we again try -5, which is a divisor of -30, we shall find it to be a root of this equation; and the next lower equation will be

$$x^2-10x-6=0;$$

of which the roots are $5\pm\sqrt{31}$. The roots of the given equation therefore are -5, -5, $5\pm\sqrt{31}$.

Ex. 3. Find the roots of the equation

$$x^4-30x^3+869x-30=0.$$

Ans. -5, 6, $\frac{1}{2}(29\pm\sqrt{837})$.

Ex. 4. Find the roots of the equation -1, 3 -3, -4, 7

$$x^5-2x^4-40x^3-10x^2+279x+252=0.$$

Ex. 5. Find the roots of the equation 1, -2 + 4, 4, $\frac{-1\pm\sqrt{5}}{2}$

$$x^6-8x^5+6x^4+61x^3-60x^2-116x+80=0.$$

Ex. 6. Find the roots of the equation 1, 1, -1, 3, -3, 2, -2

$$x^7-x^6-14x^5+14x^4+49x^3-49x^2-36x+36=0.$$

Ex. 7. Find the roots of the equation 5, -8, $\frac{-1+\sqrt{-3}}{2}$, $\frac{-1-\sqrt{-3}}{2}$

$$x^4+4x^3-36x^2-37x-40=0.$$

Ex. 8. Find the roots of the equation

$$x^5-32x^3+6x^2+175x-150=0.$$ 1, 1, -3, 5, -5

Solution of Equations by Methods of Approximation.

571. It will generally be best, to determine the *rational* roots of numerical equations by the preceding method. But rational and irrational roots may all be found with sufficient exactness by successive *approximations*. The following methods are most commonly employed for this purpose.

Approximation by Double Position.

572. From the laws of the co-efficients, as stated in Art. 541, a general estimate may be formed of the values of the roots of any equation. They must be such, that, when their signs are changed, their *product* shall be equal to the *last* term of the equation, and their *sum* equal to the co-efficient of the *second* term. A trial may then be made, by substituting, in the place of the unknown letter, its supposed value. If this proves to be too small or too great, it may be increased or diminished, and the trials repeated, till one is found which will nearly satisfy the conditions of the equations. After we

have discovered or assumed two approximate values, and calculated the errors which result from them, we may obtain a more exact correction of the root, by the following *proportion.*

As the difference of the errors, to the difference of the assumed numbers;

So is the least error, to the correction required, in the corresponding assumed number.

This is founded on the supposition, that the errors in the *results* are proportioned to the errors in the *assumed numbers.*

Let N and n be the assumed numbers;

S and s, the errors of these numbers;

R and r, the errors in the results.

Then by the supposition, $R:r::S:s$

And subt. the consequents (Art. 397,) $R-r:S-s::r:s$.

But the difference of the assumed numbers is the same as the difference of their errors. If for instance, the true number is 10, and the assumed numbers 12 and 15, the errors are 2 and 5; and the difference between 2 and 5 is the same as between 12 and 15. Substituting, then, $N-n$ for $S-s$, we have $R-r:N-n::r:s$, which is the proportion stated above.

The term *difference* is to be understood here, as it is commonly used in algebra, to express the result of subtraction according to the general rule. (Art. 75.) In this sense, the difference of two numbers, one of which is positive and the other negative, is the same as their *sum* would be, if their signs were alike. (Art. 78.)

The supposition which is made the foundation of the rule for finding the true value of the root of an equation, is not strictly correct. The errors in the results are not *exactly* proportioned to the errors in the assumed numbers. But as a greater error in the assumed number, will generally lead to a greater error in the result, than a less one, the rule will answer the purpose of approximation. If the value which is first found, is not sufficiently correct, this may be taken as one of the numbers for a second trial; and the process may be repeated till the error is diminished as much as is required. There will generally be an advantage in assuming two numbers whose difference is ·1, or ·01, or ·001, &c.

Ex. 1. Find the value of x, in the cubic equation

$$x^3-8x^2+17x-10=0.$$

Here as the signs of the terms are alternately positive and negative, the roots must be all positive; (Art. 542,) their product must be 10 and their sum 8.

Let it be supposed that one of them is 5·1 or 5·2. Then, substituting these numbers for x, in the given equation, we have

By the 1st suppos'n, $(5{\cdot}1)^3-8\times(5{\cdot}1)^2+17\times(5{\cdot}1)-10=1{\cdot}271$

By the second, $(5{\cdot}2)^3-8\times(5{\cdot}2)^2+17\times(5{\cdot}2)-10=2{\cdot}688$

That is,		By the first supposition,	By the second supposition,
The 1st term,	$x^3=$	132·651	140·608
The 2d,	$-8x^2=$	$-208{\cdot}08$	$-216{\cdot}32$
The 3d,	$17x=$	86·7	88·4
The 4th,	$-10=$	$-$ 10·	$-$ 10·
Sums or errors,		$+1{\cdot}271$	$+2{\cdot}688$
Subtracting one from the other,			1·271
Their difference is			1·417

Then stating the proportion

$$1{\cdot}4 : 0{\cdot}1 :: 1{\cdot}27 : 0{\cdot}09,$$

the correction to be subtracted from the first assumed number 5·1: The remainder is 5·01, which is a near value of x.

To correct this farther, assume $x=5{\cdot}01$, or 5·02.

		By the first supposition.	By the second supposition.
The 1st term,	$x^3=$	125·751	126·506
The 2d,	$-8x^2=$	$-200{\cdot}8$	$-201{\cdot}6$
The 3d,	$17x=$	85·17	85·34
The 4th,	$-10=$	$-$ 10·	$-$ 10·
Errors,		$+$ 0·121	$+$ 0·246
			0·121
		Difference,	0·125

Then 0·125 : 0·01 : : 0·121 : 0·01, the correction. This subtracted from 5·01, leaves 5 for the value of x; which will be found, on trial, to satisfy the conditions of the equation.

For $\quad 5^3-8\times5^2+17\times5-10=0.$

We have thus obtained one of the three roots. To find the other two, let the equation be divided by $x-5$, according to Art. 126, and it will be depressed to the next inferior degree. (Art. 538.)

$$x-5)x^3-8x^2+17x-10(x^2-3x+2=0.$$

Here the equation becomes quadratic.

By transposition, $x^2-3x=-2$

Completing the square, $x^2-3x+\frac{9}{4}=\frac{9}{4}-2=\frac{1}{4}$

Extracting and transposing, $x=\frac{3}{2}\pm\sqrt{\frac{1}{4}}=\frac{3}{2}\pm\frac{1}{2}$.

The first of these values of x, is 2, and the other 1.

We have now found the three roots of the proposed equation. When their signs are changed, their sum is -8, the co-efficient of the second term, and their product -10, the last term.

2. What are the roots of the equation

$x^3-8x^2+4x+48=0$? Ans. -2, $+4$, $+6$.

3. What are the roots of the equation

$x^3-16x^2+65x-50=0$? Ans. 1, 5, 10.

4. What are the roots of the equation

$x^3+2x^2-33x=90$? Ans. 6, -5, -3.

5. What is a near value of one of the roots of the equation

$x^3+9x^2+4x=80$?

6. What is a near value of one of the roots of the equation

$x^3+x^2+x=100$?

Newton's Method.

573. A second method of approximating to the roots of numerical equations, is that of Newton, by *successive substitutions.*

Let r be put for a number found by trial to be *nearly* equal to the root required, and let z denote the difference between r and the true root x. Then in the given equation, substitute $r\pm z$ for x, and reject the terms which contain the powers of z.

This will reduce the equation to a *simple* one. And if z be less than a unit, its powers will be still less, and therefore the error occasioned by the rejection of the terms in which they are contained, will be comparatively small. If the value of z, as found by the reduction of the new equation, be added to or subtracted from r, according as the latter is found by trial to be too great or too small, the assumed root will be once corrected.

By repeating the process, and substituting the corrected value of r, for its assumed value, we may come nearer and nearer to the root required.

Ex. 1. Find one of the values of x, in the equation

$$x^3-16x^2+65x=50.$$

Let $r-z=x$.

$$\text{Then}\left\{\begin{array}{rcccl} x^3 & = & (r-z)^3 & = & r^3-3r^2z+3rz^2-z^3 \\ -16x^2 & = & -16(r-z)^2 & = & -16r^2+32rz-16z^2 \\ 65x & = & 65(r-z) & = & 65r \quad -65z \end{array}\right\}=50.$$

Rejecting the terms which contain z^2 and z^3, we have

$$r^3-16r^2+65r-3r^2z+32rz-65z=50.$$

This reduced gives $z=\dfrac{50-r^3+16r^2-65r}{-3r^2+32r-65}$.

If r be assumed $=11$, then $z=\dfrac{60}{76}=0{\cdot}8$ nearly.

and $x=r-z$ nearly $=11-0{\cdot}8=10{\cdot}2$.

To obtain a *nearer* approximation to the root, let the corrected value of $10{\cdot}2$ be now substituted for r, in the preceding equation, instead of the assumed value 11, and we shall have

$$z={\cdot}188 \qquad\qquad x=r-z=10{\cdot}012.$$

For a *third* approximation, let $r=10{\cdot}012$, and we have

$$z={\cdot}012 \qquad\qquad x=r-z=10.$$

2. What is a near value of one of the roots of the equation
[illegible] $x^3+10x^2+5x=2600$? Ans. $11{\cdot}0067$.
3. What are the roots of the equation [illegible]
$$x^3+2x^2-11x=12?$$
4. What are the roots of the equation [illegible]
$$x^4+4x^3-7x^2-34x=24?$$

Horner's Method.

574. Another convenient method of approximating to the roots of equations was first published by Mr. Horner in 1819.

To explain this, we will suppose the given equation to be

$$2x^4-5x^3+3x^2-x-4=0.$$

By Sturm's Theorem, or by trial, according to Art. 553, we easily find that 2 is the first figure of one of its roots. To find the remaining figures, let the equation be transformed into another, whose roots are less than those of the given one by 2. The result will be

$$2y^4+11y^3+21y^2+15y-2=0. \quad \text{(See Art. 548, Ex. 1.)}$$

As y is less than x by 2, it is the decimal part of x; and the first figure of y is the second figure of x. By Art. 553, we find this figure to be .1. Let us now transform the last equation into another whose roots are less than its own by .1. The new equation will be

$2z^4+11.8z^3+24.42z^2+19.538z-.2788=0.$ (Art. 548, Ex. 2.)

The first figure of its root we shall find to be .01. This therefore is the second figure in the value of y, and the third in that of x. By proceeding thus, we can determine the root of the given equation to any required number of figures.

This method would be very inconvenient, if we were obliged to find all the figures of the required root by trial. But after obtaining one or two of them in this way, the others may be found in a much simpler manner. Thus the equation containing z may be written in the form

$$z=\frac{.2788}{19.538+24.42z+11.8z^2+2z^3}$$

Now the value of y was found to be between .1 and .2. Therefore z, which equals $y-.1$, must be less than .1; and the higher powers of z are still smaller. All the terms after 19.538, in the above denominator, are therefore small, and the value of the whole fraction can not differ much from $\frac{.2788}{19.538}$, that is .0142. Hence, z must be very nearly equal to the latter quantity. And as we require only the first figure in the value of z, we may assume this to be .01, without danger of error.

If other equations were obtained by transformation, their roots would be continually decreasing, and the first figure of each might be found, as in the case just considered, by dividing the final term of the equation, taken with a contrary sign, by the co-efficient of the last term but one.

From what has now been shown, we derive the following rule for obtaining a root of an equation of any degree.

Find by Sturm's Theorem or by trial, the first figure of the root.

Transform the equation into another whose roots are less than those of the former, by the figure just found.

Divide the last term of the new equation, taken with a contrary sign, by the co-efficient of the term before the last; to obtain a second figure of the required root.

Then transform the second equation into another whose roots are less than its own, by the figure last found:

Divide as before, to obtain a third figure: and so proceed till the required number of figures have been obtained.

It will sometimes be necessary to find the second figure of the root, as well as the first, by trial. In Ex. 3 below, even the third figure must be found in this way.

The manner of applying the rule will be shown by a few examples. And we will begin with the equation employed above.

Ex. 1. Find a root of the equation

$$2x^4-5x^3+3x^2-x-4=0.$$

The first figure of the required root, as found by trial, is 2. To find other figures, we proceed as follows.

2	-5	3	-1	-4	(2.114
	-1	1	1	$_1-2$	
	3	7	$_1 15$	$_2-.2788$	
	7	$_1 21$	17.212	$_3-.08096618$	
	$_1 11.2$	22.12	$_2 19.538$		
	11.4	23.26	19.783382		
	11.6	$_2 24.42$	$_3 20.029948$		
	$_2 11.82$	24.5382			
	11.84	24.6566			
	11.86	$_3 24.7752$			
	$_3 11.88$				

Here the co-efficients of the equation, obtained by transforming the given one as in Art. 548, Ex. 1, are indicated by the small figure $_1$, placed at the left of each; with the exception of the first co-efficient 2, which is the same as that of the primitive equation. Dividing 2, that is, -2 taken with a contrary sign, by 15, we obtain .1 for the second figure of the required root. Then from the co-efficients marked $_1$, we obtain those of the next equation, which are marked $_2$. (See Art. 548, Ex. 2.) Again, dividing .2788 by 19.538, we find .01 for the third figure of the root. And finally, by obtaining the co-efficients of the next equation, and dividing as before, we find .004 for the fourth figure of the root.

It must be observed that the mark $_1$ in the first column of numbers, applies not to the whole of the number 11.2, but to 11 only. For the sake of brevity, the decimal .2 is added at the right of 11, to form the following number 11.2; instead of writing the two numbers separately. In the same column, the mark $_2$ applies only to the first three figures of the number 11.82.

Ex. 2. Find a root of the equation

$$x^3-4x^2-4x+20=0.$$

One of its roots is between 3 and 4. (Art. 553.) For obtaining this, the process is as follows.

1	-4	-4	20 (3.525
	-1	-7	$_1-$ 1
	2	$_1-1$	$_2-$ .125
	$_1 5.5$	1.75	$_3-$ .027392
	6.0	$_2$ 4.75	$_4-$ .002171875
	$_2 6.52$	4.8804	
	6.54	$_3$ 5.0112	
	$_3 6.565$	5.044025	
	6.570	$_4$ 5.076875	
	$_4 6.575$		

The first four figures of the root are therefore 3.525.

575. When several figures of a root have been computed as in the preceding examples, a number of others may be obtained, by merely dividing the last number in the final column of the work by the last number in the previous column.

The method by which we obtained the first four figures of the root in the last example, if pursued further, would give .000427 for the next three figures. The first figure .0004 is obtained, according to the rule above, by dividing .002171875 by 5.076875. But, if we continue the division, we shall obtain the three figures .000427. Now to show that this division ought to give two or three figures of the root correctly, we observe that the equation whose co-efficients are marked $_4$, when completed by putting some letter as y for the unknown quantity, is

$$y^3+6.575y^2+5.076875y-.002171875=0.$$

And by transposition,

$$5.076875y=.002171875-6.575y^2-y^3.$$

Now, y represents the part which is to be added to the number 3.525 already found, to give the whole root of the primitive equation. (Art. 574.) And the value of y is less than .0005; for its first figure, obtained by dividing .00217 &c. by 5.07 &c. is .0004. Therefore the term $6.575y^2$ is less than .0000016 &c. and y^3 is much less than this. Hence, the preceding term .002171875 can not be affected,

by the subtraction of the other two, to the amount of more than one or two units in the sixth decimal place. If then we neglect the terms containing y^2 and y^3, and divide .002171875 by 5.076875, till the dividend, as far as the sixth decimal place, is exhausted, the quotient .000427 will, for so many figures, be the true value of y.

A greater number of figures of the root might have been found correctly by dividing in this way, if a greater number had been obtained by the previous calculation.

576. If an equation has two or more real roots, they may all be found by the rule above. Or, when one has been obtained, the given equation may be reduced (Art. 538,) to an equation of a lower degree, and another root deduced from this.

When an equation has been depressed to a quadratic, the two roots of this may be found by the rule in Art. 322. We have seen above, that 3.525427 is one of the roots of the equation

$$x^3-4x^2-4x+20=0.$$

If we divide the equation by $x-3.525427$, we obtain the quadratic

$$x^2-.474573x-5.6730716=0;$$

whence $x=.2372865\pm\sqrt{5.7293757}$; that is, x equals 2.630898, or -2.156325. Hence the three roots of the above cubic equation are $\begin{cases} 3.525427 \\ 2.630898 \\ -2.156325 \end{cases}$

By observing that the quadratic multiplied by $x-3.525427$, ought to produce the cubic equation, we shall see that the term -5.6730716, may be found by simply dividing 20 by -3.525427; and that the co-efficient $-.474573$ is equal to -4, the co-efficient of the second term in the cubic equation, diminished by -3.525427.

577. If a negative root is required, we may change the signs of the alternate terms in the proposed equation, and find the equal positive root. (Art. 550.)

Ex. 3. Find a root of the equation

$$x^3-5x+6=0.$$

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This equation has a negative root between -2 and -3. (Art. 553.) Putting ± 0 for the absent term, and changing the signs of the alternate terms, we have the new equation

$$x^3 \pm 0 - 5x - 6 = 0;$$

which has a positive root, equal to the negative root of the original equation (Art. 550.) This positive root is found as follows.

1	0	-5	-6 (2.689095
	2	-1	$_{1}-8$
	4	$_{1}7$	$_{2}-1424$
	$_{1}66$	1096	$_{3}-151168$
	72	$_{2}1528$	$_{4}-1591231$
	$_{2}788$	159104	$_{5}-88870986571$
	796	$_{3}165472$	$_{6}-5402709467625$
	$_{3}8049$	16619641	
	8058	$_{4}16692163$	
	$_{4}806709$	166928890381	
	806718	$_{5}166936150843$	
	$_{5}8067275$	16693655420675	
	8067280	$_{6}16693695757075$	

Now by dividing the last number in the final column by the last in the previous column, we may find correctly six or seven other figures of the root, namely 3236377.

Hence, one of the roots of the original equation is -2.6890953236377. The other two will be found to be imaginary.

The work may often, as in this example, be considerably abridged, by omitting the decimal point and the ciphers at the left hand of the decimal numbers in the final column or elsewhere. But care must be taken that none of the figures lose their proper local values.

Ex. 4. Find a root of the equation

$$9x^3 + 70x^2 + 500x - 547000 = 0.$$

Having found by trial (Art. 553,) that the equation has a root between 30 and 40, we proceed as follows.

9	70	500	−547000 (36.448
	340	10700	−226000
	610	29000	−18376
	880	34604	−1995904
	934	40532	−339422144
	988	4095024	−7721193472
	1042	4136992	
	10456	414120464	
	10492	414541872	
	10528	41462618816	
	105316	41471051008	
	105352		
	105388		
	1053952		
	1054024		

By dividing now, as in previous examples, we find six other figures of the root, namely 186182. Therefore

$$x=36.448186182.$$

Ex. 5. Find a root of the equation

$$x^3-48261145359159368=0;$$

in other words, extract the cube root of 48261145359159368.

The required root is between 300000 and 400000; and is found as follows.

1	0	0	−48261145359159368(364082
	3	9	−21261
	6	27	−1605145
	96	3276	−32601359159
	102	3888	−795329847368
	1084	393136	0
	1088	397488	
	109208	3975753664	
	109216	3976627392	
	1092242	397664923684	Ans. $x=364082$.

Most of the numbers in the above calculation have for convenience been shortened by omitting ciphers at the right. Thus, in the second line, 3 stands for 300000, 9 for 90000000000, and 21261 for 21261000000000000.

We may observe that the above root might have been obtained, by first finding the root of 48.261145359159368, and then multiplying the result by 100000.

Ex. 6. Find a root of the equation

$$x^4+x^3-8x-15=0.$$

Having found by trial that 2 is the first figure of one of the roots, we proceed as follows.

<table>
<tr><td>1</td><td>0</td><td>1</td><td>−8</td><td>−15 (2.302775</td></tr>
<tr><td></td><td>2</td><td>5</td><td>2</td><td>−11</td></tr>
<tr><td></td><td>4</td><td>13</td><td>28</td><td>−1259</td></tr>
<tr><td></td><td>6</td><td>25</td><td>36247</td><td>−35232966384</td></tr>
<tr><td></td><td>83</td><td>2749</td><td>45268</td><td>−3437544263|2559</td></tr>
<tr><td></td><td>86</td><td>3007</td><td>45333516808</td><td>−256233658|94300559</td></tr>
<tr><td></td><td>89</td><td>3274</td><td>4539907043|2</td><td>−28984881|244124609375</td></tr>
<tr><td></td><td>9202</td><td>32758404</td><td>4542203160|1063</td><td></td></tr>
<tr><td></td><td>920|4</td><td>3277681|2</td><td>454449972|82732</td><td></td></tr>
<tr><td></td><td>92|06</td><td>327952|24</td><td>454472943|47327063</td><td>$\dfrac{28984881 \text{ \&c.}}{45449919 \text{ \&c.}}=.6377823$</td></tr>
<tr><td></td><td>9|2087</td><td>328016|7009</td><td>45449591|457055732</td><td></td></tr>
<tr><td></td><td>92094</td><td>32808|11667</td><td>45449755|539776196125</td><td></td></tr>
<tr><td></td><td>92101</td><td>3281|456374</td><td>45449919|622726937500</td><td></td></tr>
<tr><td></td><td>921087</td><td>3281|52085009</td><td></td><td></td></tr>
<tr><td></td><td>921094</td><td>328|158532667</td><td></td><td></td></tr>
<tr><td></td><td>921101</td><td>32|8164980374</td><td></td><td></td></tr>
<tr><td></td><td>9211085</td><td>32|816544092825</td><td></td><td></td></tr>
<tr><td></td><td>9211090</td><td>32|816590148275</td><td></td><td></td></tr>
</table>

Ans. $x=2.3027756377323$.

It will be seen, by examining the calculation above, that all the figures at the right of the vertical lines, and all in the first column below the horizontal line, might have been omitted, without affecting the result. The work, as thus abridged, is exhibited below.

1	0	1	-8	$-15(2.302775$
	2	5	2	${}_{1}-11$
	4	13	${}_{1}28$	${}_{2}-1259$
	6	${}_{1}25$	36247	${}_{3}-35232966384$
	${}_{1}8$	2749	${}_{2}45268$	${}_{4}-3437544263$
	83	3007	45333516808	${}_{5}-256233658$
	86	${}_{2}3274$	${}_{3}4539907043$	${}_{6}-28984881$
	89	32758404	4542203160	
	${}_{2}92$	3277681	${}_{4}454449972$	$\frac{28984881}{45449919}=.6377323$
	9202	${}_{3}327952$	454472943	
	920	328016	${}_{5}45449591$	
	92	32808	45449755	
	${}_{3}9$	${}_{4}3281$	${}_{6}45449919$	$x=2.3027756377323$
	9	3281		
		328		
		${}_{5}32$		
		32		
		32		

In the former case, the number 83 in the first column stood for the two numbers 8 and 83, and the number 9202 for 92 and 9202. (See remark under Ex. 1.) Here, all the four numbers are distinctly given.

By attending to the above operation, it will be seen that, in each column the contraction begins in the second number below the one marked ${}_{2}$, and that the numbers go on diminishing in extent by one figure on the right, except that each of the numbers marked ${}_{3}$, ${}_{4}$, ${}_{5}$, &c. extends only as far to the right as the one below it, and the last numbers of the second and third columns have the same extent as those above them.

If we had made the calculation at first in the abridged form as above, the final figures of several of the numbers would have been different from those we have obtained by shortening the numbers which the complete calculation had furnished. Thus, the ninth number in the second column, obtained from the first seven figures of the eighth by the addition of 2×920, would be 3277680 instead of 3277681. And the seventh number in the third column, obtained from the one above it by omitting the final figure and adding 2×3277680. would have 0 for its last figure, instead of 3. But these slight differences would not affect more than one or two of the final figures in the value of x. (See remark under Ex. 7.)

In different cases, we may begin to abridge the work at different periods, according to the degree of accuracy required in the result.

Ex. 7. Find a root of the equation

$$x^3-6x+2=0.$$

Let the required root be that which lies between 2 and 3. We then proceed as follows.

1	0	-6	2(2.261802	
	2	-2	$_1-2$	
	4	$_1 6$	$_2-552$	
	$_1 6$	724	$_3-16824$	
	62	$_2 852$	$_4-7494419$	
	64	89196	$_5-20986968$	
	$_2 66$	$_3 93228$	$_6-2292506$	
	666	9329581		
	672	$_4 9336363$		
	$_3 678$	934179004		
	6781	$_5 93472177$	$\frac{2292506}{9347245}=.245260$	
	6782	9347231		
	$_4 6783$	$_6 9347245$		
	67838			
	6784			
	$_5 678$			
	6			
	6			Ans. $x=2.261802245260$.

Here we begin to abridge the work, by reducing the number next to 67838 in the first column from 67846 to 6784. We then multiply 6784 by 8, and add the product to 93417900 in the second column. This gives 93472172; which might be put for the next number in the same column. But it will be more accurate to add to this, the amount which would have been carried for tens in multiplying 67846 by 8, if the last figure had not been cut off. Now 6 multiplied by 8 gives 48; and this may be increased to 52, by taking account of the last figure of the number 934179004. If then for 52, we carry 5, we obtain the number 93472177, as given above. In a similar manner the final figures of other numbers are determined.

Ex. 8. Extract the fifth root of 123456789; or, find a root of the equation

$$x^5-123456789=0.$$

The required root is evidently between 40 and 50; and may be found as follows,

1	0	0	0	0	−123456789(41.52436
	4	16	64	256	−21056789
	8	48	256	1280	−7600588
	12	96	640	13456201	−36176890625
	16	160	656201	14128805	−6486836874
	20	16201	672604	144776381875	−541957398
	201	16403	689210	14830725312	−95999085
	202	16606	697666375	14845026875	−6805875
	203	16810	70617425	1485933533	
	204	1691275	7147337	1486219869	
	205	170157	7150781	148650623	
	2055	17119	715423	148652771	$\frac{6805875}{14865578}=.4578278$
	206	1722	71577	14865492	
	20	1722	71584	14865535	
	2	172	7159	14865578	
		17	716		
		1	716		
		1	71		
			7		
			7		

Ans. $x=41.524364578278.$

We may rely upon the accuracy of all but one or two of the last figures in this value of x.

Find the roots of the following equations.

Ex. 9. $x^3+5x^2+7x-47=0.$

Ans. $x=2.1238314040872486.$

Ex. 10. $x^3+3x^2+5x-1881718027578170.433=0.$

Ans. $x=123456.7$

Ex. 11. $x^3-27x-36=0.$

$$\text{Ans. } x=\begin{cases} 5.765722 \\ -4.320684 \\ -1.445038 \end{cases}$$

Ex. 12. $x^3-8169.5x^2-8169.5x-8170.5=0.$

$$\text{Ans. } x=\begin{cases} 8170.5 \\ -\frac{1}{2}+\frac{1}{2}\sqrt{-3} \\ -\frac{1}{2}-\frac{1}{2}\sqrt{-3} \end{cases}$$

Ex. 13. $x^3 - 484476471864 = 0.$

Ex. 14. $x^3 + x^2 + x - 13099751099 = 0.$

Ex. 15. $x^4 + 5x^3 + 4x^2 + 3x - 105 = 0.$

Ans. One value of x is 2.21733882735297.

Ex. 16. $x^4 - 80x^3 + 1998x^2 - 14937x + 5000 = 0.$

Ans. One value of x is $.3509870458$.

Ex. 17. $x^4 - 19x^3 + 123x^2 - 302x + 200 = 0.$

$$\text{Ans. } x = \begin{cases} 1.02804 \\ 4.00000 \\ 6.57653 \\ 7.39543 \end{cases}$$

Ex. 18. $x^5 - 7x^4 + 15x^3 - 58x^2 + 44x - 300 = 0.$

Ans. $x = 6.119538$.

SECTION XXI.

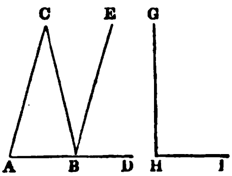
APPLICATION OF ALGEBRA TO GEOMETRY.*

ART. **578.** IT is often expedient to make use of the algebraic notation for expressing the relations of geometrical quantities, and to throw the several steps in a demonstration into the form of equations. By this, the nature of the reasoning is not altered. It is only translated into a different *language*. *Signs* are substituted for *words*, but they are intended to convey the same meaning. A great part of the demonstrations in Euclid, really consist of a series of equations, though they may not be presented to us under the algebraic forms. Thus the proposition, that *the sum of the three angles of a triangle is equal to two right angles*, (Euc. 32, 1,) may be demonstrated, either in common language, or by means of the signs used in Algebra.

* This Section is to be read *after* the Elements of Geometry.

Let the side *AB*, of the triangle *ABC*, (Fig. 1,) be continued to *D*; let the line *BE* be parallel to *AC*; and let *GHI* be a right angle.

Fig. 1.



The demonstration, in words, is as follows;

1. The angle *EBD* is *equal* to the angle *BAC*, (Euc. 29, 1.)
2. The angle *CBE* is *equal* to the angle *ACB*.
3. Therefore, the angle *EBD added* to *CBE*, that is, the angle *CBD*, is *equal* to *BAC added* to *ACB*.
4. If to these equals, we add the angle *ABC*, the angle *CBD added* to *ABC*, is *equal* to *BAC added* to *ACB* and *ABC*.
5. But *CBD added* to *ABC*, is *equal* to twice *GHI*, that is, to two right angles. (Euc. 13, 1.)
6. Therefore, the angles *BAC*, and *ACB*, and *ABC*, are together equal to twice *GHI*, or two right angles.

Now by substituting the sign +, for the word *added*, or *and*, and the character =, for the word *equal*, we shall have the same demonstration in the following form.

1. By Euclid 29, 1, $EBD=BAC$
2. And $CBE=ACB$
3. Add the two equations $EBD+CBE=BAC+ACB$
4. Add ABC to both sides $CBD+ABC=BAC+ACB+ABC$
5. But by Euclid 13, 1, $CBD+ABC=2GHI$
6. Make the 4th & 5th equal $BAC+ACB+ABC=2GHI$.

By comparing, one by one, the steps of these two demonstrations, it will be seen, that they are precisely the same, except that they are differently expressed. The algebraic mode has often the advantage, not only in being more *concise* than the other, but in exhibiting the *order* of the quantities more distinctly to the eye. Thus, in the fourth and fifth steps of the preceding example, as the parts to be compared are placed one under the other, it is seen, at once, what must be the new equation derived from these two. This regular arrangement is very important, when the demonstration of a theorem, or the resolution of a problem, is unusually complicated. In ordinary language, the numerous relations of the

quantities, require a series of explanations to make them understood; while by the algebraic notation, the whole may be placed distinctly before us, at a single view. The disposition of the men on a chess-board, or the situation of the objects in a landscape, may be better comprehended, by a glance of the eye, than by the most labored description in words.

579. It will be observed, that the notation in the example just given differs, in one respect, from that which is generally used in algebra. Each quantity is represented, not by a *single letter*, but by *several*. In common algebra, when one letter stands immediately before another, as ab, without any character between them, they are to be considered as *multiplied* together.

But in geometry, AB is an expression for a *single line*, and not for the product of A into B. Multiplication is denoted, either by a point or by the character $\times$. The product of AB into CD, is $AB.CD$, or $AB\times CD$.

580. There is no impropriety, however, in representing a geometrical quantity by a single letter. We may make b stand for a line or an angle, as well as for a number.

If, in the example above, we put the angle

$EBD=a$,	$ACB=d$,	$ABC=h$,
$BAC=b$,	$CBD=g$,	$GHI=l$;
$CBE=c$,		

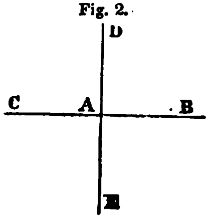
the demonstration will stand thus;

1. By Euclid 29, 1,	$a=b$
2. And	$c=d$
3. Adding the two equations,	$a+c=g=b+d$
4. Adding h to both sides,	$g+h=b+d+h$
5. By Euclid 13, 1,	$g+h=2l$
6. Making the 4th and 5th equal,	$b+d+h=2l$.

This notation is, apparently, more simple than the other; but it deprives us of what is of great importance in geometrical demonstrations, a continual and easy reference to the figure. To distinguish the two methods, *capitals* are generally used, for that which is peculiar to geometry; and *small letters*, for that which is properly algebraic. The latter has the advantage in long and complicated processes, but the other is often to be preferred, on account of the facility with which the figures are consulted.

581. If a line, whose length is measured from a given point or line, be considered *positive*; a line proceeding in the *opposite* direction is to be considered *negative*. If *AB* (Fig. 2,) reckoned from *DE* on the *right*, is positive; *AC* on the *left* is negative.

Fig. 2.



A line may be conceived to be produced by the *motion of a point.* Suppose a point to move in the direction of *AB*, and to describe a line varying in length with the distance of the point from *A*. While the point is moving towards *B*, its distance from *A* will *increase.* But if it move from *B* towards *C*, its distance from *A* will *diminish,* till it is reduced to nothing, and then will increase on the *opposite side.* As that which increases the distance on the right, diminishes it on the left, the one is considered positive, and the other negative. See Arts. 54, 55.

Hence, if in the course of a calculation, the algebraic value of a line is found to be *negative;* it must be measured in a direction opposite to that which, in the same process, has been considered positive.

582. In algebraic calculations, there is frequent occasion for *multiplication, division,* involution, &c. But how, it may be asked, can *geometrical* quantities be multiplied into each other? One of the factors, in multiplication, is always to be considered as a *number.* (Art. 86.) The operation consists in repeating the multiplicand as many times as there are *units* in the multiplier. How then can a *line,* a *surface,* or a *solid,* become a multiplier?

To explain this, it will be necessary to observe, that whenever one geometrical quantity is multiplied into another, some *particular extent* is to be considered *the unit.* It is immaterial what this extent is, provided it remains the same, in different parts of the same calculation. It may be an inch, a foot, a rod, or a mile. If an *inch* is taken for the unit, each of the lines to be multiplied, is to be considered as made up of so many parts, as it contains inches. The multiplicand will then be repeated, as many times, as there are units in the multiplier. If, for instance, one of the lines be a foot long, and the other half a foot; the factors will be, one 12 inches, and the other 6, and the product will be 72 inches. Though it would be absurd to say that one line is to be re-

peated *as often as another is long;* yet there is no impropriety in saying, that one is to be repeated as many times, as there are feet or rods in the other. This, the nature of a calculation often requires.

583. If the line which is to be the multiplier, is only a *part* of the length taken for the unit; the product is a like part of the multiplicand. (Art. 85.) Thus, if one of the factors is 6 inches, and the other half an inch, the product is 3 inches.

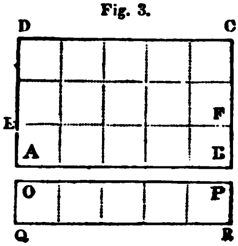
584. Instead of referring to the measures in common use, as inches, feet, &c. it is often convenient to fix upon one of the lines in a figure, as the unit with which to compare all the others. When there are a number of lines drawn within and about a *circle*, the *radius* is commonly taken for the unit. This is particularly the case in trigonometrical calculations.

585. The observations which have been made concerning lines, may be applied to *surfaces* and *solids*. There may be occasion to multiply the *area* of a figure, by the number of inches in some given line.

But here another difficulty presents itself. The product of two lines is often spoken of, as being equal to a *surface;* and the product of a line and a surface, as equal to a *solid.* Thus the area of a parallelogram is said to be equal to the product of its base and height; and the solid contents of a cylinder, are said to be equal to the product of its length into the area of one of its ends. But if a line has no *breadth,* how can the multiplication, that is the *repetition,* of a line produce a surface? And if a surface has no *thickness,* how can a repetition of it produce a solid?

If a parallelogram, represented on a reduced scale by $ABCD$, (Fig. 3,) be five inches long, and three inches wide; the area or surface is said to be equal to the product of 5 into 3, that is, to the number of inches in AB, multiplied by the number in BC. But the inches in the lines AB and BC are *linear* inches, that is, inches in *length* only; while those which compose the surface AC are *superficial* or *square* inches, a different species of magnitude. How can one of these be converted

Fig. 3.



into the other by multiplication, a process which consists in repeating quantities, without changing their nature?

586. In answering these inquiries, it must be admitted, that measures of length do not belong to the same class of magnitudes with superficial or solid measures; and that none of the steps of a calculation can, properly speaking, transform the one into the other. But, though a line can not become a surface or a solid, yet the several measuring units in common use are so adapted to each other, that squares, cubes, &c. are bounded by lines of the same name. Thus the side of a square inch, is a linear inch; that of a square rod, a linear rod, &c. The *length* of a linear inch is, therefore, the same as the length or breadth of a square inch.

If then several square inches are placed together, as from *Q* to *R*, (Fig. 3,) the *number* of them in the parallelogram *OR* is the same as the number of linear inches in the side *QR*: and if we know the length of this, we have of course the area of the parallelogram, which is here supposed to be one inch wide.

But, if the breadth is *several* inches, the larger parallelogram contains as many smaller ones, each an inch wide, as there are inches in the whole breadth. Thus, if the parallelogram *AC* (Fig. 3,) is 5 inches long, and 3 inches broad, it may be divided into three such parallelograms as *OR*. To obtain, then, the number of squares in the large parallelogram, we have only to multiply the number of squares in one of the small parallelograms, into the number of such parallelograms contained in the whole figure. But the number of square inches in one of the small parallelograms is equal to the number of linear inches in the *length AB*. And the number of small parallelograms, is equal to the number of linear inches in the *breadth BC*. It is therefore said concisely, that the *area of the parallelogram is equal to the length multiplied into the breadth.*

587. We hence obtain a convenient algebraic expression, for the area of a right-angled parallelogram. If two of the sides perpendicular to each other are *AB* and *BC*, the expression for the area is $AB\times BC$; that is, putting a for the area,

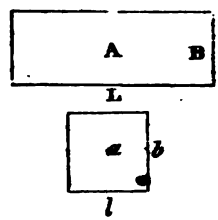
$$a=AB\times BC.$$

It must be understood, however, that when *AB* stands for a *line*, it contains only *linear* measuring units; but when it enters into the expression for the *area*, it is supposed to con-

tain *superficial* units of the same name. Yet as, in a given length, the *number* of one is equal to that of the other, they may be represented by the same letters, without leading to error in calculation.

588. The expression for the area may be derived, by a method more simple, but less satisfactory perhaps to some, from the principles which have been stated concerning *variable quantities*, in the 12th section. Let a (Fig. 4,) represent a square inch, foot, rod, or other measuring unit; and let b and l be two of its sides. Also, let A be the area of any right-angled parallelogram, B its breadth, and L its length. Then it is evident, that, if the breadth of each were the same, the areas would be as the lengths; and, if the length of each were the same, the areas would be as the breadths.

Fig. 4.



That is, $A : a :: L : l$, when the breadth is given;

And $A : a :: B : b$, when the length is given;

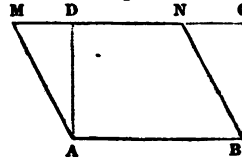
Therefore, (Art. 430,) $A : a :: B \times L : bl$, when both vary.

That is, the area is as the *product* of the *length* and *breadth*.

589. Hence, in quoting the Elements of Euclid, the term *product* is frequently substituted for *rectangle*. And whatever is there proved concerning the equality of certain rectangles, may be applied to the product of the lines which contain the rectangles.*

590. The area of an *oblique* parallelogram is also obtained by multiplying the base into the perpendicular height. Thus the expression for the area of the parallelogram $ABNM$ (Fig. 5,) is $MN \times AD$ or $AB \times BC$. For by Art. 587, $AB \times BC$ is the area of the right-angled parallelogram $ABCD$; and by Euclid 36, 1,† parallelograms upon equal bases, and between the same parallels, are equal; that is, $ABCD$ is equal to $ABNM$.

Fig. 5.



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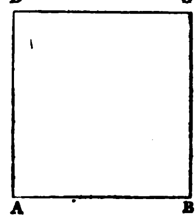
* See Note Q.

† Legendre's Geometry, American Edition, Art. 166.

591. The area of a *square* is obtained, by multiplying one of the sides *into itself*. Thus the expression for the area of the square AC, (Fig. 6,) is $\overline{AB}^2$, that is,

Fig. 6.



$$a=\overline{AB}^2.$$

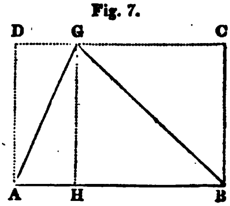
For the area is equal to $AB \times BC$. (Art. 587.)

But $AB=BC$, therefore,

$$AB \times BC = AB \times AB = \overline{AB}^2.$$

592. The area of a *triangle* is equal to *half* the product of the base and height. Thus the area of the triangle ABG, (Fig. 7,) is equal to half AB into GH or its equal BC, that is,

Fig. 7.

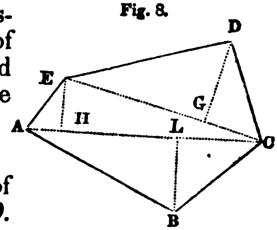


$$a=\tfrac{1}{2}AB \times BC.$$

For the area of the parallelogram $ABCD$ is $AB \times BC$, (Art. 587.) And by Euclid 41, 1,* if a parallelogram and a triangle are upon the same base, and between the same parallels, the triangle is half the parallelogram.

593. Hence, an algebraic expression may be obtained for the area of any figure whatever, which is bounded by right lines. For every such figure may be divided into triangles.

Fig. 8.



Thus the right-lined figure $ABCDE$ (Fig. 8,) is composed of the triangles ABC, ACE, and ECD.

The area of the triangle	$ABC=\frac{1}{2}AC \times BL$,
That of the triangle	$ACE=\frac{1}{2}AC \times EH$,
That of the triangle	$ECD=\frac{1}{2}EC \times DG$.

The area of the whole figure is, therefore, equal to

$$(\tfrac{1}{2}AC \times BL)+(\tfrac{1}{2}AC \times EH)+(\tfrac{1}{2}EC \times DG).$$

The explanations in the preceding articles contain the first principles of *the mensuration of superficies*. The object of

* Legendre, 168.

introducing the subject in this place, however, is not to make a practical application of it, at present; but merely to show the grounds of the method of representing geometrical quantities in algebraic language.

594. The expression for the superficies has here been derived from that of a line or lines. It is frequently necessary to *reverse* this order; to find a side of a figure, from knowing its area.

If the number of square inches in the parallelogram $ABCD$ (Fig. 3,) whose breadth BC is 3 inches, be divided by 3; the quotient will be a parallelogram $ABFE$, *one* inch wide, and of the same length with the larger one. But the length of the small parallelogram, is the length of its side AB. The number of *square* inches in one is the same, as the number of *linear* inches in the other. (Art. 586.) If therefore, the area of the large parallelogram be represented by a, the side $AB=\frac{a}{BC}$, that is, *the length of a parallelogram is found by dividing the area by the breadth.*

595. If a be put for the area of a *square* whose side is AB,

Then by Art. 591, $\qquad a=\overline{AB}^2$

And extracting both sides, $\qquad \sqrt{a}=AB$

That is, *the side of the square is found, by extracting the square root of the number of measuring units in its area.*

596. If AB be the base of a *triangle* and BC its perpendicular height;

Then by Art. 592, $\qquad a=\frac{1}{2}BC\times AB$

And dividing by $\frac{1}{2}BC$, $\qquad \frac{a}{\frac{1}{2}BC}=AB.$

That is, *the base of a triangle is found, by dividing the area by half the height.*

597. As a *surface* is expressed, by the product of its length and breadth; the contents of a *solid* may be expressed, by the product of its length, breadth and depth. It is necessary to bear in mind, that the measuring unit of solids, is a *cube;* and that the side of a cubic inch, is a square inch; the side of a cubic foot, a square foot, &c.

Let $ABCD$ (Fig. 3,) represent the base of a parallelopiped, five inches long, three inches broad, and *one* inch deep. It is evident there must be as many *cubic* inches in the solid, as there are *square* inches in its base. And, as the product of the lines AB and BC gives the area of this base, it gives, of course, the contents of the solid. But suppose that the depth of the parallelopiped, instead of being *one* inch, is *four* inches. Its contents must be four times as great. If, then, the length be AB, the breadth BC, and the depth CO, the expression for the solid contents will be,

$$AB \times BC \times CO.$$

598. By means of the algebraic notation, a geometrical demonstration may often be rendered much more simple and concise than in ordinary language. The proposition, (Euc. 4, 2,) that when a straight line is divided into two parts, the square of the whole line is equal to the squares of the two parts, together with twice the product of the parts, is demonstrated, by involving a binomial.

Let the side of a square be represented by s;

And let it be divided into two parts, a and b.

By the supposition, $\quad s = a + b$

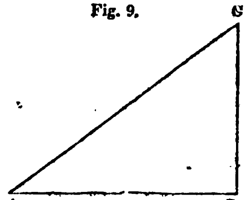
And squaring both sides, $\quad s^2 = a^2 + 2ab + b^2.$

That is, s^2 the square of the whole line, is equal to a^2 and b^2, the squares of the two parts, together with $2ab$, twice the product of the parts.

599. The algebraic notation may also be applied, with great advantage, to the solution of geometrical *problems.* In doing this, it will be necessary, in the first place, to raise an algebraic equation, from the geometrical relations of the quantities given and required; and then by the usual reductions, to find the value of the unknown quantity in this equation. See Art. 195.

Prob. 1. Given the *base,* and the *sum* of the hypothenuse and perpendicular, of the right angled triangle, ABC, (Fig. 9,) to find the perpendicular.

Fig. 9.



$$\left.\begin{array}{ll}\text{Let the base} & AB=b \\ \text{The perpendicular} & BC=x \\ \text{The sum of hyp. \& per.} & x+AC=a \\ \text{Then transposing } x, & AC=a-x\end{array}\right\}$$

1. By Euclid 47, 1,* $\overline{BC}^2+\overline{AB}^2=\overline{AC}^2$

2. That is, by the notation, $x^2+b^2=(a-x)^2=a^2-2ax+x^2$.

Here we have a common algebraic equation, containing only one unknown quantity. The reduction of this equation in the usual manner, will give

$$x=\frac{a^2-b^2}{2a}=BC, \text{ the side required.}$$

The solution, in letters, will be the same for any right angled triangle whatever, and may be expressed in a general theorem, thus; 'In a right angled triangle, the perpendicular is equal to the square of the sum of the hypothenuse and perpendicular, diminished by the square of the base, and divided by twice the sum of the hypothenuse and perpendicular.'

It is applied to particular cases by substituting *numbers*, for the letters a and b. Thus if the base is 8 feet, and the sum of the hypothenuse and perpendicular 16, the expression $\frac{a^2-b^2}{2a}$ becomes $\frac{16^2-8^2}{2\times16}=6$, the perpendicular; and this subtracted from 16, the sum of the hypothenuse and perpendicular, leaves 10, the length of the hypothenuse.

Prob. 2. Given the *base* and the *difference* of the hypothenuse and perpendicular, of a right angled triangle, to find the perpendicular.

Let the base AB (Fig. 10,) $=b=20$
The perpendicular, $BC=x$
The given difference, $=d=10$
Then will the hypoth. $AC=x+d$.

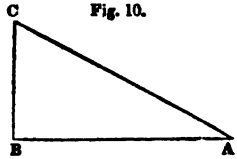


Fig. 10.

Then

1. By Euclid 47, 1, $\overline{AC}^2=\overline{AB}^2+\overline{BC}^2$
2. That is, by the notation, $(x+d)^2=b^2+x^2$
3. Expanding $(x+d)^2$, $x^2+2dx+d^2=b^2+x^2$
4. Therefore, $x=\dfrac{b^2-d^2}{2d}=15$.

Prob. 3. If the hypothenuse of a right angled triangle is 30 feet, and the difference of the other two sides 6 feet, what is the length of the base? Ans. 24 feet.

* Legendre, 186.

Prob. 4. If the hypothenuse of a right angled triangle is 50 rods, and the base is to the perpendicular as 4 to 3, what is the length of the perpendicular? Ans. 30.

Prob. 5. Having the perimeter and the diagonal of a parallelogram $ABCD$, (Fig. 11,) to find the sides.

$$\left.\begin{array}{ll}\text{Let the diagonal} & AC=h=10\\ \text{The side} & AB=x\\ \text{Half the perimeter} & BC+AB=BC+x=b=14\\ \text{Then by transposing } x, & BC=b-x\end{array}\right\}$$

Fig. 11.

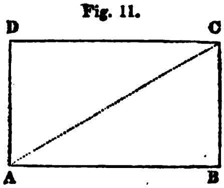
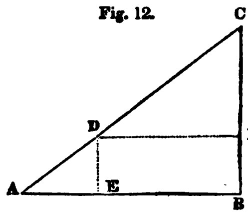


Fig. 12.



By Euclid 47, 1, $\overline{AB}^2+\overline{BC}^2=\overline{AC}^2$.

That is, $x^2+(b-x)^2=h^2$

Therefore, $x=\frac{1}{2}b\pm\sqrt{\frac{1}{2}h^2-\frac{1}{4}b^2}=8$

Here the side AB is found; and the side BC is equal to $b-x=14-8=6$.

Prob. 6. The area of a right angled triangle ABC (Fig. 12,) being given, and the sides of a parallelogram inscribed in it, to find the side BC.

$$\left.\begin{array}{ll}\text{Let the given area } =a, & DE=BF=b\\ EB=DF=d, & BC=x\\ \text{Then by the figure,} & CF=BC-BF=x-b.\end{array}\right\}$$

1. By similar triangles, $CF:DF::BC:AB$
2. That is, $x-b:d::x:AB$
3. Therefore, $dx=(x-b)\times AB$
4. By Art. 592, $a=AB\times\frac{1}{2}BC=AB\times\frac{1}{2}x$
5. Dividing by $\frac{1}{2}x$, $\frac{2a}{x}=AB$
6. Therefore, $dx=(x-b)\times\frac{2a}{x}=2a-\frac{2ab}{x}$
7. And $x=\frac{a}{d}\pm\sqrt{\frac{a^2}{d^2}-\frac{2ab}{d}}=BC.$

Prob. 7. The three sides of a right angled triangle, *ABC*, (Fig. 13,) being given, to find the segments made by a perpendicular, drawn from the right angle to the hypothenuse.

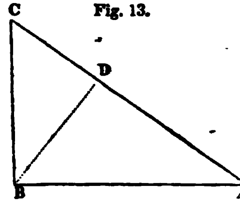


Fig. 13.

The perpendicular will divide the original triangle, into two right angled triangles, *BCD* and *ABD*. (Euc. 8, 6.)

1. By Euc. 47, 1, $\overline{\mathrm{BD}}^2+\overline{\mathrm{CD}}^2=\overline{BC}^2$
2. By the figure, $\mathrm{CD}=AC-\mathrm{AD}$
3. Squaring both sides, $\overline{\mathrm{CD}}^2=(AC-\mathrm{AD})^2$
4. Therefore, $\overline{\mathrm{BD}}^2+(AC-\mathrm{AD})^2=\overline{BC}^2$
5. Expanding, $\overline{\mathrm{BD}}^2+\overline{AC}^2-2AC.\mathrm{AD}+\overline{\mathrm{AD}}^2=\overline{BC}^2$
6. Transposing, $\overline{\mathrm{BD}}^2=\overline{BC}^2-\overline{AC}^2+2AC.\mathrm{AD}-\overline{\mathrm{AD}}^2$
7. By Euc. 47, 1, $\overline{\mathrm{BD}}^2=\overline{AB}^2-\overline{\mathrm{AD}}^2$
8. Making 6th & 7th eq. $\overline{BC}^2-\overline{AC}^2+2AC.\mathrm{AD}=\overline{AB}^2$
9. Therefore, $\mathrm{AD}=\dfrac{\overline{AB}^2+\overline{AC}^2-\overline{BC}^2}{2AC}$.

The *unknown* lines, to distinguish them from those which are known, are here expressed by Roman letters.

Prob. 8. Having the area of a parallelogram *DEFG* (Fig. 14,) inscribed in a given triangle, *ABC*, to find the sides of the parallelogram.

Draw *CI* perpendicular to *AB*. By supposition, *DG* is parallel to *AB*. Therefore,

The triangle *CHG*, is similar to *CIB*
And *CDG*, to *CAB*

Let $CI=d$ $DG=x$
$AB=b$ The given area $=a$

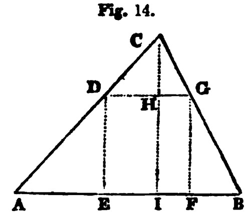


Fig. 14.

1. By similar triangles, $CB:CG::AB:DG$
2. And $CB:CG::CI:CH$
3. By equal ratios, (Art. 392,) $AB:DG::CI:CH$

4. Therefore, $\frac{DG\times CI}{AB}=CH$

5. By the figure, $CI-CH=IH=DE$

6. Substituting for CH, $CI-\frac{DG\times CI}{AB}=DE$

7. That is $d-\frac{dx}{b}=DE$

8. By Art. 587, $a=DG\times DE=x\times\left(d-\frac{dx}{b}\right)$

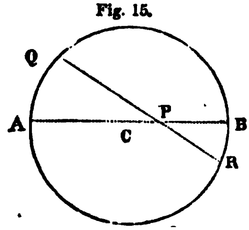
9. That is, $a=dx-\frac{dx^2}{b}$

10. This reduced gives $x=\frac{b}{2}\pm\sqrt{\left(\frac{b^2}{4}-\frac{ab}{d}\right)}=DG.$

The side DE is found, by dividing the area by DG.

Prob. 9. Through a given point, in a given circle, so to draw a right line, that its parts, between the point and the periphery, shall have a given difference.

Fig. 15.



In the circle $AQBR$, (Fig. 15,) let P be a given point in the diameter AB.

Let $AP=a$, $PR=x$,

$BP=b$, The given difference $=d$,

Then will $PQ=x+d$.

1. By Euc. 35, 3,* $PR\times PQ=AP\times BP$
2. That is, $x\times(x+d)=a\times b$
3. Or, $x^2+dx=ab$
4. Completing the square, $x^2+dx+\frac{1}{4}d^2=\frac{1}{4}d^2+ab$
5. Extracting and transposing, $x=-\frac{1}{2}d\pm\sqrt{\frac{1}{4}d^2+ab}=PR.$

With a little practice, the learner may very much abridge these solutions, and others of a similar nature, by reducing several steps to one.

Prob. 10. If the sum of two of the sides of a triangle be 1155, the length of a perpendicular drawn from the angle included between these to the third side be 300, and the

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* Legendre, 224.

difference of the segments made by the perpendicular, be 495; what are the lengths of the three sides?

Ans. 945, 375, and 780.

Prob. 11. If the perimeter of a right angled triangle be 720, and the perpendicular falling from the right angle on the hypothenuse be 144; what are the lengths of the sides?

Ans. 300, 240, and 180.

Prob. 12. The difference between the diagonal of a square and one of its sides being given, to find the length of the sides.

If $x=$ the side required, and $d=$ the given difference;

$$\text{Then}\quad x=d+d\sqrt{2}.$$

Prob. 13. The base and perpendicular height of any plane triangle being given, to find the side of a square inscribed in the triangle, and standing on the base, in the same manner as the parallelogram *DEFG*, on the base *AB*, (Fig. 14.)

If $x=$ a side of the square, $b=$ the base, and $h=$ the height of the triangle;

$$\text{Then}\quad x=\frac{bh}{b+h}.$$

Prob. 14. Two sides of a triangle, and a line bisecting the included angle being given; to find the length of the base or third side, upon which the bisecting line falls.

If $x=$ the base, $a=$ one of the given sides, $c=$ the other, and $b=$ the bisecting line;

$$\text{Then}\quad x=(a+c)\times\sqrt{\frac{ac-b^2}{ac}}.$$

Prob. 15. If the hypothenuse of a right angled triangle be 35, and the side of a square inscribed in it, in the same manner as the parallelogram *BEDF*, (Fig. 12,) be 12; what are the lengths of the other two sides of the triangle?

Ans. 28, and 21.

Prob. 16. The number of feet in the perimeter of a right angled triangle, is equal to the number of square feet in the area; and the base is to the perpendicular as 4 to 3. Required the length of each of the sides.

Ans. 6, 8, and 10.

Prob. 17. A grass plat 12 rods by 18, is surrounded by a gravel walk of uniform breadth, whose area is equal to that of the grass plat. What is the breadth of the gravel walk?

Prob. 18. The sides of a rectangular field are in the ratio of 6 to 5; and one-sixth of the area is 125 square rods. What are the lengths of the sides?

Prob. 19. There is a right angled triangle, the area of which is to the area of a given parallelogram as 5 to 8. The shorter side of each is 60 rods, and the other side of the triangle adjacent to the right angle, is equal to the diagonal of the parallelogram. Required the area of each?

Ans. 4800 and 3000 square rods.

Prob. 20. There are two rectangular vats, the greater of which contains 20 cubic feet more than the other. Their capacities are in the ratio of 4 to 5; and their bases are squares, a side of each of which is equal to the depth of the other vat. Required the depth of each?

Ans. 4 and 5 feet.

Prob. 21. Given the lengths of three perpendiculars, drawn from a certain point in an equilateral triangle, to the three sides; to find the length of the sides.

If a, b, and c, be the three perpendiculars, and $x=$ half the length of one of the sides;

$$\text{Then}\quad x=\frac{a+b+c}{\sqrt{3}}.$$

Prob. 22. A square public green is surrounded by a street of uniform breadth. The side of the square is 3 rods less than 9 times the breadth of the street; and the number of square rods in the street, exceeds the number of rods in the perimeter of the square by 228. What is the area of the square?

Ans. 576 rods.

Prob. 23. Given the lengths of two lines drawn from the acute angles of a right angled triangle, to the middle of the opposite sides; to find the lengths of the sides.

If $x=$ half the base, $y=$ half the perpendicular, and a and b equal the two given lines;

$$\text{Then}\quad x=\sqrt{\frac{4b^2-a^2}{15}}\qquad y=\sqrt{\frac{4a^2-b^2}{15}}.^*$$

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* See Note R.

NOTES.

Note A. Page 1.

As the term *quantity* is here used to signify whatever is the object of mathematical inquiry, it will be obvious that *number* is meant to be included; so far at least, as it can be the subject of mathematical investigation. Dugald Stewart asserts, indeed, that it might be easily shown, that number does not fall under the definition of quantity in *any* sense of that word.* For proof that it is included in the *common* acceptation of the word, it will be sufficient to refer to almost any mathematical work in which the term quantity is explained, and particularly to the familiar distinction between *continued* quantity or magnitude, and *discrete* quantity or number.

But does number "fall under the *definition* of quantity?" Mr. Stewart after quoting the observation of Dr. Reid, that the object of the mathematics is commonly said to be quantity, which ought to be defined, *that which may be measured*, adds, "The appropriate objects of this science are such things alone as admit not only of being increased and diminished, but of being *multiplied and divided.* In other words, the common character which characterizes all of them, is their *mensurability.*" That number may be multiplied and divided, will not probably be questioned. But it may perhaps be doubted, whether it is capable of mensuration. If, as Mr. Locke observes, "number is that which the mind makes use of, in measuring all things that are measurable," can it measure *itself*, or be measured? It is evident that it can not be measured *geometrically*, by applying to it a measure of length or capacity. But by measuring a quantity mathematically, what else is meant, than determining the *ratio* which it bears to some other quantity of the same kind; in other words finding how often one is contained in the other, either exactly or with a certain excess? And is not this as applicable to number as to magnitude? The ratio

* Philosophy of the Mind, Vol. II, Note G.

which a given number bears to *unity* can not, indeed, be the subject of *inquiry;* because it is expressed by the number itself. But the ratio which it bears to *other* numbers may be as proper an object of mathematical investigation, as the ratio of a mile to a furlong.

For proof that number is not quantity, Mr. Stewart refers to Barrow's Mathematical Lectures. Dr. Barrow has started an *etymological* objection to the application of the term quantity to number, which he intimates might, with more propriety, be called *quotity.* He observes, "The *general object* of the mathematics has no proper name, either in Greek or Latin." And adds, "It is plain the mathematics is conversant about two things especially, quantity strictly taken, and quotity; or magnitude and multitude." There is frequent occasion for a common name, to express number, duration, &c. as well as magnitude; and the term quantity will probably be used for this purpose, till some other word is substituted in its stead.

But though Dr. Barrow thus distinguishes between magnitude and number, he afterwards gives it as his opinion, (page 20, 49,) that there is really no quantity in nature different from what is called magnitude or continued quantity, and consequently that this alone *ought to be accounted the object of the mathematics.* He accordingly devotes a whole lecture to the purpose of proving the *identity of arithmetic and geometry.* (Lect. 3.) He is "convinced that number really differs nothing from what is called continued quantity; but is only formed to express and declare it;" that as "the conceptions of magnitude and number could scarcely be separated," by the ancients, "in the *name,* they can hardly be so in the *mind,*" and "that number includes in it every consideration pertaining to geometry." He admits of *metaphysical* number, which is not the object of geometry, or even of the mathematics. But, in his view, magnitude is always included in *mathematical* number, as the units of which it is composed are *equal.* On the other hand, magnitudes are not to be considered as mathematical quantities, except as they are measured by number. In short, *quantity is magnitude measured by number.*

It would seem, then, that according to Dr. Barrow, number considered as separate from magnitude, has as fair a claim to be called quantity, as magnitude considered as separate from number. If arithmetic and geometry are the *same;*

quantity is as much the object of one, as of the other. How far this scheme is applicable to duration, motion, &c. it is not necessary, in this place, to inquire.

Note B. p. 35.

It is common to define multiplication, by saying that 'it is finding a product which has the same ratio to the multiplicand, that the multiplier has to a unit.' This is strictly and universally true. But the objection to it, *as a definition*, is, that the idea of ratio, as the term is understood in arithmetic and algebra, seems to imply a previous knowledge of multiplication, as well as of division. In this work at least, the expression of geometrical ratio is made to depend on division, and division on multiplication. Ratio, therefore, could not be properly introduced into the definition of multiplication.

It is thought, by some, to be absurd to speak of a *unit* as consisting of *parts*. But whatever may be true with respect to number *in the abstract*, there is certainly no absurdity in considering an integer, of one denomination, as made up of parts of a different denomination. *One* rod may contain several feet: *one* foot several inches, &c. And in multiplication, we may be required to repeat the whole, or a part of the multiplicand, as many times as there are inches in a foot, or part of a foot.

Note C. p. 55.

Strictly speaking, the inquiry to be made is, how often the *whole* divisor is contained in as many terms of the dividend. But it is easier to divide by a *part* only of the divisor; and this will lead to no error in the result, as the whole divisor is multiplied, in obtaining the several subtrahends.

Note D. p. 86.

It is perhaps more philosophically exact, to consider an equation as affirming the equivalence of two different expressions of the same quantity, than to speak of it as expressing an equality between one quantity and another. But it is doubted whether the former definition is the best adapted to the apprehension of the learner; who in this early part of his mathematical course, may be supposed to be very little accustomed to abstraction. Though he may see clearly, that the area of a triangle is *equal* to the area of a parallelogram

of the same base and half the height; yet he may hesitate in pronouncing that the two surfaces are precisely the *same*.

Note E. p. 129.

As the *direct* powers of an integral quantity have *positive* indices, while the *reciprocal* powers have *negative* indices; it is common to call the former *positive powers*, and the latter *negative powers*. But this language is ambiguous, and may lead to mistake. For the same terms are applied to powers with positive and negative signs *prefixed*. Thus $+8a^4$ is called a positive power; while $-8a^4$ is called a negative one. It may occasion perplexity, to speak of the latter as being both positive and negative at the same time; positive, because it has a positive *index*, and negative because it has a negative co-efficient. This ambiguity may be avoided, by using the terms direct and reciprocal; meaning, by the former, powers with positive exponents, and by the latter, powers with negative exponents.

Note F. p. 202.

For the sake of keeping clear of the multiplied controversies, a great portion of them verbal, respecting the nature of ratio, I have chosen to define geometrical ratio to be that which is *expressed* by the quotient of one quantity divided by another, rather than to say that it *consists* in this quotient. Every ratio which can be mathematically assigned, may be expressed in this way, if we include surd quantities among those which are to be admitted into the numerator or denominator of the fraction representing the quotient.

Note G. p. 204.

This definition of compound ratio is more comprehensive than the one which is given in Euclid. That is included in this, but is limited to a particular case, which is stated in Art. 357. It may answer the purposes of geometry, but is not sufficiently general for algebra.

Note H. p. 206.

It is not denied that very respectable writers use these terms indiscriminately. But it appears to be without any necessity. The ratio of 6 to 2 is 3. There is certainly a

difference between *twice* this ratio, and the *square* of it, that is, between twice three, and the square of three. All are agreed to call the latter a *duplicate* ratio. What occasion is there, then, to apply to it the term *double* also? This is wanted, to distinguish the other ratio. And if it is confined to that, it is used according to the common acceptation of the word, in familiar language.

NOTE I. p. 214.

The definition here given is meant to be applicable to quantities of every description. The subject of proportion as it is treated of in Euclid, is embarrassed by the means which are taken to provide for the case of *incommensurable* quantities. But this difficulty is avoided by the algebraic notation which may represent the ratio even of incommensurables.

Thus the ratio of 1 to $\sqrt{2}$ is $\frac{1}{\sqrt{2}}$.

It is impossible, indeed, to express in rational numbers, the square root of 2, or the ratio which it bears to 1. But this is not necessary, for the purpose of showing its equality with another ratio.

The product $4\times2=8$.

And, as equal quantities have equal roots,

$2\times\sqrt{2}=\sqrt{8}$, therefore, $2:\sqrt{8}::1:\sqrt{2}$.

Here the ratio of 2 to $\sqrt{8}$, is proved to be the same, as that of 1 to $\sqrt{2}$; although we are unable to find the exact value either of $\sqrt{8}$ or $\sqrt{2}$.

It is impossible to determine, with perfect accuracy, the ratio which the side of a square has to its diagonal. Yet it is easy to prove, that the side of one square has the *same* ratio to its diagonal, which the side of any other square has to its diagonal. When incommensurable quantities are once reduced to a proportion, they are subject to the same laws as other proportionals. Throughout the section on proportion, the demonstrations do not imply that we know the *value* of the terms, or their ratios; but only that one of the ratios is *equal* to the other.

Note K. p. 218.

The inversion of the means can be made with strict propriety in those cases only in which all the terms are quantities of the same kind. For, if the two last be different from the two first, the antecedent of each couplet, after the inversion will be different from the consequent, and therefore, there can be no ratio between them. (Art. 359.)

This distinction, however, is of little importance in practice. For, when the several quantities are expressed in *numbers*, there will always be a ratio between the numbers. And when two of them are to be multiplied together, it is immaterial which is the multiplier, and which is the multiplicand. Thus in the Rule of Three in arithmetic, a change in the order of the two middle terms will make no difference in the result.

Note L. p. 225.

The terms *composition* and *division* are derived from geometry, and are introduced here, because they are generally used by writers on proportion. But they are calculated rather to perplex, than to assist the learner. The objection to the word *composition* is, that its meaning is liable to be mistaken for the composition or compounding of *ratios*. (Art. 398.) The two cases are entirely different, and ought to be carefully distinguished. In one, the terms are *added*, in the other, they are *multiplied* together. The word compound has a similar ambiguity in other parts of the mathematics. The expression $a+b$, in which a is *added* to b, is called a compound quantity. The fraction $\frac{1}{2}$ of $\frac{2}{3}$, or $\frac{1}{2}\times\frac{2}{3}$, in which $\frac{1}{2}$ is *multiplied* into $\frac{2}{3}$, is called a compound fraction.

The term *division*, as it is used here, is also exceptionable. The alteration to which it is applied, is effected by *subtraction*, and has nothing of the nature of what is called division in arithmetic and algebra. But there is another case, (Art. 400,) totally distinct from this, in which the change in the terms of the proportion is actually produced by division.

Note M. p. 234.

The principles stated in this section, are not only expressed in different language, from the corresponding propositions in Euclid, but are in several instances more general. Thus the first proposition, in the fifth book of the Elements, is confined

to equimultiples. But the article referred to, as containing this proposition, is applicable to all cases of equal *ratios*, whether the antecedents are multiples of the consequents or not.

Note N. p. 250.

The solution of one of the cases is omitted in the text, because it is performed by *logarithms*, with which the learner is supposed not to be acquainted, in this part of the course. When the first term, the last term, and the ratio are given, the *number* of terms may be found by the formula

$$n=\frac{\log. \frac{rz}{a}}{\log. r}.$$

Note O. p. 255.

When it is said that a mathematical quantity may be supposed to be increased beyond any determinate limits, it is not intended that a quantity can be specified so great, that no limits greater than this can be assigned. The quantity and the limits may be *alternately* extended one beyond the other. If a line be conceived to reach the most distant point in the visible heavens, a limit may be mentioned beyond this. The line may then be supposed to be extended farther than this limit. Another point may be specified still farther on, and yet the line may be conceived to be carried beyond it.

Note P. p. 257.

The apparent *contradictions* respecting infinity, are owing to the ambiguity of the term. It is often thought that the proposition, that quantity is infinitely divisible, involves an absurdity. If it can be proved that a line an *inch* long can be divided into an infinite number of parts, it can, by the same mode of reasoning, be proved, that a line *two inches* long may be first divided in the middle, and then *each* of the sections be divided into an infinite number of parts. In this way, we shall obtain one infinite *twice as great* as another.

If by infinity, here is meant that which is beyond any assignable limits, one of these infinites may be supposed greater than the other, without any absurdity. But if it be meant that the number of divisions is so great that it can not be increased, we do not prove this, concerning *either* of the

lines. We make out, therefore no contradiction. The apparent absurdity arises from shifting the meaning of the terms. We demonstrate that a quantity is, in one sense infinite; and then infer that it is infinite, in a sense widely different.

Note Q. p. 386.

It will be thought, perhaps, that it was unnecessary to be so particular, in obtaining the expression for the area of a parallelogram, for the use of those who read Playfair's edition of Euclid, in which "*AD.DC* is put for the rectangle contained by *AD* and *DC*." It is to be observed, however, that he introduces this, merely as an article of *notation*. (Book II, Def. 1.) And though a point interposed between the letters, is, in algebra, a sign of multiplication; yet he does not here undertake to show how the sides of a parallelogram may be multiplied together. In the first book of the *Supplement*, he has indeed demonstrated, that "equiangular parallelograms are to one another, as the products of the numbers proportional to their sides." But he has not given to the expressions the forms most convenient for the succeeding parts of this work. In making the transition from pure geometry to algebraic solutions and demonstrations, it is important to have it clearly seen that the geometrical principles are not altered; but are only expressed in a different language.

Note R. p. 395.

This section comprises very little of what is commonly understood by the application of algebra to geometry. The principal object has been, to prepare the way for the other parts of the course, by stating the grounds of the algebraic notation of geometrical quantities, and rendering it familiar by a few examples.

[illegible]

$x =$ [illegible]

$x + 10 =$ [illegible]

$$12x = x + 10$$

$$11x = 11$$

$$x = \frac{11}{11}$$

$$\frac{5.4x}{2} + \frac{x^2}{2} - 4x = 5.40 + 11x + 5.76$$

"Be thine bright-Intellect's
unfading treasures
And Piety's more deeply-holier
spell."

"Be thine bright - Intellect's
unfading brightness
And Poesy's more deeply - hallowed
spell."

Sherborn Sept 1865